CERTAIN CLASSES OF GROUP PRESENTATIONS

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A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

1993

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BY

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A thesis submitted for the degree of Doctor of Philosophy

of the University of St Andrews

Department of Mathematical and

Computational Sciences

Oct. 1992

ABSTRACT

In Chapter two we look at the class $F(n) = \langle R,S \mid R^n = S^n = (R^{a_1}S^{b_1})^{x_1}(R^{c_1}S^{d_1})^{y_1}(R^{a_2}S^{b_2})^{x_2}(R^{c_2}S^{d_2})^{y_2} \ldots (R^{a_m}S^{b_m})^{x_m}(R^{c_m}S^{d_m})^{y_m} = 1 \rangle$. For some values of n, a_i, b_i, c_i, d_i, x_i, y_i we give results on these groups where we have been able to determine their order, either finite or infinite. In the last section in Chapter two we study two classes of groups generated by A and B and subject to the following relations :

Relations for class 1:

$A^4 = 1, B^4 = 1, (B(AB)^2)^4 = 1, (B(BA)^6)^4 = 1, (B(BA)^{14})^4 = 1, \ldots,$

$(B(BA)^{(2^{(n-1)/2}-2)})^4 = 1,$

$(A^{-1}B^{-1})^{2^{(n-3)/2}}B(BA)^{(2^{(n-1)/2}-2)}B(BA)^{2^{(n-3)/2}}B^{-1}(A^{-1}B^{-1})^{(2^{(n-1)/2}-2)}B^{-1}$

$(A^{-1}B^{-1})^{(2^{(n+1)/2}-3)}A(BA)^{(2^{(n-1)/2}-1)}B^{-1} = 1,$

$(BA)^{2^{(n-1)/2}}B^{-1}(A^{-1}B^{-1})^{(2^{(n-1)/2}-2)}B^{-1}(A^{-1}B^{-1})^{(2^{(n+1)/2}-3)}A^2 = 1.$

Relations for class 2:

$A^4 = 1, B^4 = 1, (B(AB)^2)^4 = 1, (B(BA)^6)^4 = 1, (B(BA)^{14})^4 = 1, \ldots, (B(BA)^{(2^{n/2}-2)})^4 = 1, B^{-1}(BA)^{2^{(n-2)/2}}B(BA)^{(2^{n/2}-2)}B(A^{-1}B^{-1})^{(2^{(n-2)/2}-1)}A(BA)^{(2^{n/2}-1)} = 1,$

$(BA)^{(2^{n/2}+2^{(n-2)/2}+2)}B(BA)^{(2^{n/2}-2)}B(A^{-1}B^{-1})^{(2^{(n-2)/2}-1)}A^2 = 1.$

The groups in the first class turn out to be the cyclic group of order 2 and the groups in the second class turn out to be metabelian groups of order $4.(2^{n/2}-1)^2$. Moreover the derived group of the groups in the second class is the direct product of two copies of a cyclic group of order $(2^{n/2}-1)^2$. In Chapter three we study the groups with a presentation of the form:

$$\langle A,B \mid A^4 = 1, B^n = 1, A^iB^jA^kB^t = 1$$

and determine all possibilities with conditions: $j+t = 0$ and $i,k \in \{ \mp 1, 2 \}$.

Also in the second section of Chapter three we study the groups with a presentation of the form:

$$\langle A,B \mid A^4 = 1, B^n = 1, A^iB^jA^kB^tA^mB^p = 1 \rangle$$

and determine some of the possibilities with conditions:

$j = 1$, $t = 1$, $p = -2$ and $i,k,m \in \mathbb{Z}$. In Chapter four we give new efficient presentations for the groups PSL(2,p), where p is an odd prime, $p \in \{ 5,7,11,13,17,19,23,29,31,37, 41,43,53,59,79,83,89,109,139,229 \}$. We give permutation generators for these groups which satisfy our efficient presentation. Also we give new efficient presentations for PSL(2,p), where p is a prime power and $p \in \{ 9,25,27,49,169 \}$. Also in Chapter four, permutation generators are given for these groups which satisfy our presentations. In Chapter five we give new efficient presentations for the groups SL(2,p), where p is an odd prime and $p \in \{ 5,7,11,13,17,19,23,29,31,41, 43,53,79,89,109,139,229 \}$. Also we give new efficient presentations for the groups SL(2,p), where p is an prime power and $p \in \{ 8,16,25,27,49,169 \}$. In Chapter six we study the class of groups with the presentation

$\langle a,b \mid a^p = 1,\ b^{m+p}a^{-m}b^{m}a^{-m} = 1,\ (ab)^2 = 1 \rangle$, p an odd number and $m \in \mathbb{Z}$. For some values of p and m these groups have connections with the groups PSL(2,p).

In Chapter 7 we attempt to show the efficiency of $PSL(2,\mathbb{Z}_n) \times PSL(2,\mathbb{Z}_m)$. For some values of n and m we give efficient presentation for these groups. In the same chapter we also attempt to show the efficiency of $PSL(2,\mathbb{Z}_p) \times PSL(2,3^2)$. For some values of p we give an efficient presentation for these groups. In the last section of the thesis we give efficient presentations for the following direct products

(i) $PSL(2,5) \times PSL(2,3^2)$

(ii) $PSL(2,7) \times PSL(2,3^2)$

(iii) $PSL(2,5) \times PSL(2,3^3)$

Also in the last section of the thesis the structure of a perfect group of order 161280 is investigated.

DECLARATIONS

I, Bilal Vatansever, hereby certify that this thesis has been composed by myself, that it is a record of my own work, and that it has not been accepted in any previous application for a higher degree.

Signed .... Date 23 Oct 1992

I was admitted to the Faculty of Science of The University of St. Andrews under Ordinance General No. 12 on 10/10/1988 and as a candidate for the degree of Ph.D. on 10/10/1989.

Signed .... Date 23 Oct 1992

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PREFACE

I am most deeply indebted to my supervisor Dr E.F. Robertson, under whose supervision this work has been carried out; for his constant encouragement and invaluable guidance, for introducing me to research in computational group theory, and for his great help in reading and correcting the manuscript.

I would like to express my gratitude to Dr C.M. Campbell for his inspiring and informative lectures which I attended during my stay in St. Andrews as well for his constructive suggestions.

My thanks is also to David Gill for many helpful conversations.

I am extremely grateful to the Cukurova University for the financial support over the last four years.

This thesis is dedicated to my parents whose constant encouragement and love has inspired me in pursuing my studies abroad.

Finally, I would like also to dedicate this thesis to many friends of mine who have given me a lot of love and friendship.

CERTIFICATION

I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate to the Degree of Ph.D.

Signature of Supervisor Date 23 Oct-1992

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION:

In this thesis in Chapters two and three we study several classes of finitely presented groups. In order to investigate the structure of the groups in these classes we show how the Todd - Coxeter coset enumeration algorithm and modified Todd - Coxeter algorithm can be used. In particular in Chapter two we look at the class $F(n) = \langle R,S \mid R^n = S^n = (R^{a_1}S^{b_1})^{x_1}(R^{c_1}S^{d_1})^{y_1}(R^{a_2}S^{b_2})^{x_2}(R^{c_2}S^{d_2})^{y_2} \ldots (R^{a_m}S^{b_m})^{x_m}(R^{c_m}S^{d_m})^{y_m} = 1 \rangle$. For some values of n, a_i, b_i, c_i, d_i, x_i, y_i we give results on these groups where we have been able to determine their order, either finite or infinite. In the last section of Chapter two we study two different classes of finitely presented groups; their presentations are given in the introduction of Chapter two.

Chapter three considers two classes of groups with two generators and three relations. Both classes have a similar presentation to groups considered in the paper [10] by C.M. Campbell and R.M.Thomas, On (2,n) - Groups related to Fibonacci Groups. However, in the presentations we consider, one generator has order 4 instead of 2. We attempt to find the order of these groups and to determine their structure.

In Chapter four we give new efficient presentations for the groups PSL(2,p), where p is an odd prime, p ∈ { 5,7,11,13,17,19,23,29,31,37,41,43,53,59,79,83,89,109,139, 229 }. We give permutation generators for these groups which satisfy our efficient presentation. Also we give new efficient presentations for PSL(2,p), where p is a prime power and p ∈ { 9,25,27,49,169 }. Also in this chapter, permutation generators are given for these groups which satisfy our presentations. In general, the efficiency of PSL(2,p), p prime, has been solved by H. Zassenhaus [43] , (see also J. Sunday [36]). Later in [7] C.M.Campbell and E.F.Robertson gave the most symmetric presentations so far obtained. Their presentation was based on a presentation due to Beetham [3] and a reduction of this presentation in [34]. But the efficiency of PSL(2,p), p a prime power has not yet been solved. There are general results although they do not give an efficient presentation. Recently C.M.Campbell, E.F.Robertson and P.D.Williams in [12] have given presentations for the groups PSL($2,p^n$), p prime, which show that the deficiency of these groups is bounded. In the same paper, in particular, they have given efficient presentations for PSL($2,3^4$), PSL($2,5^3$), PSL($2,11^2$), PSL($2,13^2$) and PSL($2,19^2$). C.M.Campbell and E.F.Robertson have also given an efficient presentation for PSL($2,3^2$). For the groups PSL($2,3^3$), PSL($2,5^2$) and PSL($2,7^2$) efficient presentation are given in [13].

In Chapter five we give new efficient presentations for the groups SL(2,p), where p is an odd prime and p ∈ { 5,7,11,13,17,19,23,29,31,41,43,53,79,89,109,139,229 } . Also we give new efficient presentations for the groups SL(2,p), where p is an prime power and p ∈ { 8,16,25,27,49,169 }. In 1980 C.M.Campbell and E.F.Robertson, in [6], gave an efficient presentation for the groups SL(2,p), p an odd prime. But in general the efficiency of SL($2,p^n$) has not yet been solved. For small values of p and n there are results in the literature. In particular in [6] C.M.Campbell and E.F.Robertson have given an efficient presentation for SL(2,8) and in [8], by the same authors, the efficiency of SL(2,16) is shown. In [9] the efficiency of SL(2,25), SL(2,27), SL(2,32), SL(2,49) and

SL(2,64) is proved. In 1988 in [11] the efficiency of SL(2,169) was shown by C.M.Campbell and E.F.Robertson.

In Chapter six we study the class of groups with the presentation $\langle\, a,b \mid a^p = 1,\ b^{m+p}a^{-m}b^m a^{-m} = 1,\ (ab)^2 = 1 \,\rangle$, p an odd number and $m \in \mathbb{Z}$. For some values of p and m these groups have connections with the groups PSL(2,p).

Questions concerning the efficiency of direct products have been of considerable interest for a number of years. In general, for every prime p, the efficiency of $PSL(2,p) \times PSL(2,p)$ has been shown in [13] by C.M.Campbell , E.F.Robertson and P.D.Williams. The efficiency of direct products of the groups $PSL(2,p^{n_i})$ for a fixed prime p and different n_i 's have been studied in [12] by C.M.Campbell , E.F.Robertson and P.D.Williams. Also in the same paper they studied the efficiency of $PSL(2,p_1) \times PSL(2,p_2)$, p_1, p_2 prime powers. In Chapter 7 we attempt to show the efficiency of $PSL(2,\mathbb{Z}_n) \times PSL(2,\mathbb{Z}_m)$. For some values of n and m we give an efficient presentation for these groups. In the same chapter we also attempt to show the efficiency of $PSL(2,\mathbb{Z}_p) \times PSL(2,3^2)$. For some values of p we give an efficient presentation for these groups. In the last section of the thesis we give efficient presentations for the following direct products

(i) $PSL(2,5) \times PSL(2,3^2)$

(ii) $PSL(2,7) \times PSL(2,3^2)$

(iii) $PSL(2,5) \times PSL(2,3^3)$

Also in the last section of the thesis the structure of a perfect group of order 161280 is investigated.

In the next section we give some terminology and notation from a few areas of group theory to be used in later chapters.

1.2 DEFINITIONS AND THEOREMS:

1.2.1 Definition: Suppose that X is a set, $F = F(X)$ is the free group on X, R is a subset of F, $N = \bar{R}$ is the normal closure of R in F and $G = F/N$. Then we write $G = \langle X \mid R \rangle$ and call this a presentation of G.

The elements of X are called generators and those of R relators.

A group G is called finitely presented if it has such a presentation with both X and R finite sets.

We shall only be concerned with finite presentations.

1.2.2 Definition: If A, B are subgroups of a group G then we shall use the notation $[A,B] = \{ [a,b] : a \in A, b \in B \}$. In particular , we call $[G,G]$ the derived group of G. We shall denote it by G'

The following theorem shows the importance of presentations; a proof is given in Johnson [24].

Theorem 1.2.1 Every group has a presentation and every finite group is finitely presented.

Also the proofs of the following two theorems are given in [24].

Theorem 1.2.2 If $G = \langle X \mid R \rangle$ and $H = \langle Y \mid S \rangle$ are two presentations, then the product $G \times H$ has a presentation $\langle X,Y \mid R, S, [X,Y] \rangle$ where $[X,Y]$ denotes the set of commutators $\{ [x,y] \mid x \in X, y \in Y \}$.

Theorem 1.2.3 If $G = \langle X \mid R \rangle$, then $G/G' = \langle X \mid R, C \rangle$, where $X = \{ x_1, x_2, .., x_r \}$ and C is the set $\{ [x_i, x_j] \mid 1 \le i < j \le r \}$.

Definition 1.2.3 Let $G = \langle X \mid R \rangle$ be a finite presentation of G. Then $|X| - |R|$ is called the deficiency of this presentation of G. The deficiency of a finitely presented group G is defined by $\operatorname{def}G = \max\{|X| - |R| : \text{all finite presentations } \langle X \mid R \rangle \text{ of } G \}$.

Theorem 1.2.4 Every finitely presented group with positive deficiency is necessarily infinite.

Definition 1.2.4 A group G is metacyclic if it has a normal subgroup N such that G/N and N are cyclic.

Definition 1.2.5 A group G is metabelian (or soluble of length two) if , and only if, G' is abelian.

Definition 1.2.6 A group G is perfect if G = G', the commutator subgroup ($|G/G'| = 1$).

Definition 1. 2.7 If G is a finite group with presentation $G = F/\bar{R}$ (F is a free group of finite rank) then the Schur multiplier M(G) is defined by

$$(F' \cap \bar{R})/[F,\bar{R}]$$

where $\bar{R}$ is the normal closure of R in F = F(X).

Theorem 1.2.5 (Schur (see [40]))

(i) M(G) is a finite abelian group.

(ii) $M(G)$ is independent of the finite presentation $\langle X \mid R \rangle$.

(iii) $M(G)$ can be generated by $-\mathrm{def}G$ elements.

Definition 1.2.8 A covering group C of a group G is a group which contains a normal subgroup A satisfying the conditions $C/A \cong G$, $A < C' \cap Z(C)$ and $|A| = |M(G)|$, where $Z(C)$ is the centre of C.

<u>**Special Linear and Projective Special Linear groups:**</u>

$SL(2,p)$ is the group of 2×2 matrices of determinant 1 over $GF(p)$, p a prime. For the commutative ring R with 1 define $SL(2,R)$ to be the group of 2×2 matrices with determinant 1 over R. Define $PSL(2,R) = SL(2,R)/\{\pm I\}$ where I is the 2×2 identity matrix. If R is the finite field $GF(p^n)$ for p a prime, we write $PSL(2,R) = PSL(2,p^n)$ and $SL(2,R) = SL(2,p^n)$. We shall also be interested in the case where $R = \mathbb{Z}_k$.

Theorem 1.2.8 (see [25]).

(i) (Steinberg 1961, 1967)

The Schur multiplier of $SL(2,p^n)$ is trivial with the following exceptions:

(a) If G is $SL(2,4)$ then $M(G) = C_2$

(b) If G is $SL(2,9)$ then $M(G) = C_3$.

(ii) If $PSL(2,p^n)$ is not one of the following groups in (d), (e) below, then

$M(PSL(2,p^n)) = C_m$, where $m = (p^n - 1, 2)$.

(d) $M(PSL(2,4)) = C_2$

(e) $M(PSL(2,9)) = C_6$.

Given two groups G_1 and G_2 then the Schur - Kunneth formula [40] asserts

$$M(G_1\times G_2) = M(G_1)\times M(G_2)\times(G_1 \otimes G_2).$$

Thus, when G_1 or G_2 is perfect, $M(G_1 \times G_2) = M(G_1) \times M(G_2)$ so the multiplier of a direct product of simple groups is the direct product of the multipliers of the simple groups.

1.3 ALGORITHMS AND PROGRAMS:

1.3.1 Todd-Coxeter Algorithm and Modified Algorithm:

E.H.Moore [30] and others have computed the index of a subgroup in an abstract group by systematically enumerating the cosets. In 1936 J.A.Todd and H.S.M.Coxeter [39] described an algorithm for enumerating the cosets of a finitely generated subgroup of finite index in a finitely presented group. This almost mechanical technique was later adapted for use on computers. An early program was written by Haselgrove at Cambridge in 1953. Many other mathematicians have used or implemented the TC algorithm. Some names in this context are Beetham, Leech, Felsch, Havas and Cannon.

Alongside this enumeration one of the useful modifications of the algorithm which is described by Coxeter and Moser [18,Chapter 2], is determining a presentation of a subgroup. We shall show how the modified algorithm may be used to give a presentation of the subgroup in terms of the given subgroup generators. For more information look at [1, 2, 4, 14, 24, 27, 29, 31].

We give a brief description of the coset enumeration method;

<u>Description of the method:</u>

Suppose we are given a finite presentation for a group G of the form

$$G = \langle\, x_1, x_2, \dots, x_n \mid r_i,\ 1 \le i \le m \,\rangle$$

where each r_i is a word in $\bar{X} = \{\, x_1, x_2, \dots, x_n, x_1^{-1}, \dots, x_n^{-1} \,\}$.

Let $h_1, h_2, \dots, h_t \in G$ where each h_i is a word in $\bar{X}$. Let H be the subgroup $\langle h_1, \dots, h_t \rangle$.

The algorithm determines [G:H] if it is finite.

First express each r_i as a product of generators x_i and their inverses such that we have no exponent other than ± 1. We set up relation tables such that each one is headed by one of the given relations. If a relation has k factors its table contains k+1 columns in which every factor occurs between two columns. We also set up a table with a single row for each subgroup generator h_i, again written out fully as words in the x_i's with powers ± 1. This table is called the subgroup table of h_i. We also set up a table which is headed by the $x_1, x_2, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}$ and has as many rows as [G:H]. This table is called the coset table. We now use positive integers to denote right cosets of H in G. The subgroup itself is denoted by 1. The initial information which we have is $1h_i = 1$, $i = 1,2, \ldots ,t$. We begin by entering 1 in the first and last places of the first row of each subgroup table and relation table. We then consider an empty space next to some 1 and fill it with the coset 2. After defining the coset 2 i.e. for some j, $1 \le j \le n$; $1x_j = 2$ (or $2x_j^{-1} = 1$) then we have $2x_j^{-1} = 1$ (or $1x_j = 2$). We record this definition $1x_j = 2$ (or $2x_j^{-1} = 1$) in the coset table. Now we put a 2 in the first and last places of the second row of each relation table, we then put this information and its consequence ($2x_j^{-1} = 1$) into the tables wherever possible. This process is known as scanning. Having made sure that no more spaces can be filled in this way, we enter the coset 3 in an empty space that is adjacent to a filled space. Having recorded the new definition (of the form $ix = 3$ where $x \in \bar{X}$) in our coset table we begin a new row in the relation tables and scan again. Similarly we can introduce the cosets 4,5, ... We continue in this way until our tables are completed. This means that there are no more empty spaces in the tables, so we have [G:H] rows in each relation table. Our definition of new cosets may not be sufficient to complete the tables. We clearly need more information of the form $kx = l$ (where $x \in \bar{X}$) than is contained in our definitions of new cosets and such information will be obtained when a row of a relation table or in the subgroup table becomes complete. Such information is called a deduction. In such deductions three possibilities can occur:

(1) The places of both kx and lx^{-1} in the coset table are still empty. In this case insert l into the place of kx and k into the place of lx^{-1}, and fill in all the relevant places in the other tables. Here we have obtained new information.

(2) The place of kx in the coset table is already filled by l (and lx^{-1} by k) therefore we get no new information.

(3) At least one of the places kx or lx^{-1} in the table is filled by a number different to that just obtained by the deduction. Here we realise that actually we have given two different numbers a,b say, to the same coset. This is called a collapse (sometimes called a coincidence). When such a collapse is found we replace the larger number by the smaller everywhere in the coset tables and in the subgroup tables. Replacing may lead to further coincidences: if further coincidences occur we deal with them in the same way as before. This completes the coset enumeration algorithm.

Some applications of the TC algorithm:

(i) If we take the trivial subgroup then the coset enumeration will give the order of the group.

(ii) The algorithm can be used to determine whether a given subgroup is normal. This can be checked as follows:

Let $H = \langle h_1, \dots , h_t \rangle$ be a subgroup of a group G and suppose that $[G:H] = q$, finite, then H is normal in G if, and only if, $ih_j = i$, $i = 1, \dots , q$; $j = 1, 2, \dots ,t$.

(iii) We may find a permutation representation for the group G.

Next we give the Modified Algorithm.

The Modified TC Algorithm:

Let $G = \langle x_1, x_2, \dots, x_n \mid r_i = 1, 1 \le i \le m \rangle$ where each r_i is a word in $\bar{X} = \{ x_1, x_2, \dots, x_n, x_1^{-1}, \dots, x_n^{-1} \}$. Let H be the subgroup $\langle h_1, h_2, \dots, h_t \rangle$.

Suppose $[G:H] = d$ (finite).

Here we will deal with coset representatives rather than cosets. We shall find a set of coset representatives ,let as say $(1= q_1), q_2, \dots, q_d$ and using this set we shall obtain a presentation for the subgroup H. We will choose 1, the identity element of G, as the coset representative of H. We use the same integer both for a coset and for its representative. By setting up the relation tables and subgroup tables and using the initial information we carry out the algorithm as before and develop tables with information of the type

$$q_i x_j = w_{ij} q_k$$

where q_i, q_k are the representatives of the cosets i and k and w_{ij} is a word in the generators of H. The tables which we form will contain information of the type

$$i x_j = w_{ij} k$$
$$k x_j^{-1} = w_{ij}^{-1}\ i$$

where i,k are the representatives of the cosets i and k.

Our initial information is $1h_i = 1 (i = 1, 2, \dots, t)$.

If a generator h_i of H is a generator of G, say $h_i = x_j$, then if we think of 1 as the coset H we have

$$1x_j = 1 \quad (1 \text{ as coset H})$$
$$1x_j = x_j.1. \quad (1 \text{ as the identity element of G})$$

The cosets from 2 upwards are always defined by $ix_j = (i+1)$ (or $ix_j^{-1} = (i+1)$). Here i is an integer and this leads to the following information.

(I) ix_j is the representative of the coset i+1, where i is the coset representative of the coset i

(II) $ix_j = 1.(i+1)$ and $(i+1)x_j^{-1} = 1.i$.

If $ix_j = k$ is new information which has been obtained by completing a row of a table, then we consider the place where new information has been won.

Let us say the new information is found from the table headed by $a_1a_2 \dots a_t \dots a_p = 1$ and is $fa_t = g$ where the a's $\in \bar{X}$, f,g are integers denoting cosets.

From $a_t = a_{t-1}^{-1} \; a_{t-2}^{-1} \dots a_1^{-1} \, a_p^{-1} \dots a_{t+1}^{-1}$

$$fa_t = f\, a_{t-1}^{-1} \; a_{t-2}^{-1} \dots a_1^{-1} \, a_p^{-1} \dots a_{t+1}^{-1}$$

By the above assumption the right hand side will be written as

$$fa_t = w_{t-1}s_1 \, a_{t-2}^{-1} \dots a_1^{-1} \, a_p^{-1} \dots a_{t+1}^{-1}$$

$$fa_t = w_{t-1}w_{t-2}s_2 \, a_{t-3}^{-1} \dots a_1^{-1} \, a_p^{-1} \dots a_{t+1}^{-1}$$

"

"

"

$$fa_t = w_{t-1}w_{t-2} \dots w_1 w_p \dots w_{t+2}s_{p-1} \, a_{t+1}^{-1}$$

and finally,

$fa_t = w_{t-1}w_{t-2} \dots w_1 w_p \dots w_{t+2}w_{t+1}g$ (from the table it is understood that $s_{p-1} \, a_{t+1}^{-1} = w_{t+1}g$)

and hence $fa_t = wg$ where $w = w_{t-1}w_{t-2} \dots w_1 w_p \dots w_{t+1}$ is a word in the subgroup H.

When closure occurs we will have a complete set of representatives and tables containing all the relations of the type

$fa_i = w_{fi}g$

where f,g are coset representatives, a_i is a generator x_j or its inverse and w_{fi} is a word in the generators of H.

However we might have a coincidence like $ix_j = w_1.k_1$ and $ix_j = w_2k_2$ where w_1 and w_2 are words in H. Suppose $k_2 > k_1$ then $k_2 = w_2^{-1}w_1k_1$. Replacing k_2 with k_1 in every occurrence of k_2 then H can be presented on its original generators $h_1, h_2, \dots, h_t$ as follows:

$H = \langle h_1, h_2, \dots, h_t \mid 1h_i = h_i.1,\ i = 1,2, \dots, t;\ jr_kj^{-1} = 1$ where $j = 1,2, \dots, d;$

$k = 1,2, \dots, m \rangle$

where $d = [G:H]$.

1.3.2 The Reidemeister - Schreier (R.S.) process:

This is a method of finding a presentation of a subgroup of finite index in a group, but the presentation is not in terms of the given subgroup generators.

Here, we give a brief outline of the method: more information can be obtained in [24, chapters 1,4], [22, chapter 7] and Magnus, Karass and Solitar [28, section 2.3].

Lexicographic ordering:

Let F be free group on X. Assume that $X = \{ x_1, x_2, \dots, x_n \}$. Let $\bar{X}$ to be ordered as $x_1 < x_2 < \dots < x_n < x_1^{-1} < \dots < x_n^{-1}$. Now we can order the elements of F as follows:

$a<b$ if $L(a)<L(b)$, $a,b \in F$ and $L(a),L(b)$ are the lengths of the words a and b respectively. For two different elements of the same length the ordering is determined by comparing elements in the same position.

i.e. $a = a_1a_2 \dots a_n$, $b = b_1b_2 \dots b_n$ and m is the least integer where $a_m \neq b_m$. Then we say

$$a<b \Leftrightarrow a_m<b_m.$$

A Schreier transversal:

Let H to be a subgroup of F(X). Now the cosets of H yield a partition of F and we can find a subset U such that, for every $x \in F$, there is exactly one $u \in U$ for which $x \in Hu$, i.e. $F = \cup Hu$ where $u \in U$. Such a subset U is called a transversal for H in F. The Schreier transversal for H in F(X) is obtained if the representative in U of each coset is taken to be the least element of that coset.

The Algorithm:

Now suppose $F = F(X)$, $G = \langle X \mid R \rangle = F/\bar{R}$,$\bar{R}$ is the normal closure of the subgroup generated by the elements of R and H is a given subgroup of G. We wish to find a presentation for H. Certainly, for a subgroup K of F we can have $H = K/\bar{R}$. Let us suppose that U is a Schreier transversal for K in F. Let $u \in U$, $x \in X$ then $ux \in F$. There exists a unique element $v \in U$ such that $ux \in Hv$ (for U is a transversal). Since v depends on u and x we denote it by $v = \underline{ux}$. We have $ux\underline{ux}^{-1} \in K$ because $ux \in Hv$ so $ux = hv$ for some $h \in H$ and then $uxv^{-1} = ux\underline{ux}^{-1} \in H$.

The set $A = \{ux\underline{ux}^{-1} \mid u \in U, x \in X$, ux is not an element of U $\}$ generates K. We can show that A generates K.

Let $x \in K$. Since $x \in F$, then $x = x_1, x_2, \ldots, x_n$ where $x_i \in X$. Define sequences of elements as follows:

$$u_1 = 1,\ u_{i+1} = \underline{u_i x_i} \quad , i \geq 1$$

$$v_i = u_i x_i u_{i+1}^{-1} \quad , 1 \leq i \leq n.$$

Obviously $v_i \in A$ so $v_1 v_2 \ldots v_n = u_1 x_1 x_2 \ldots x_n u_{n+1}^{-1} = x u_{n+1}^{-1}$ belongs to H. Since $x \in K$ then $u_{n+1} \in K$. On the other hand $u_{n+1} \in U$, hence $u_{n+1} = 1$(because 1 is the unique element of U denoting the coset K. i.e. the trivial coset). Thus, $x = x_1, x_2, \ldots, x_n$,proving that A generates K.

The condition (ux is not element of U) is inserted because of the fact that " $ux\underline{ux}^{-1} = 1$ if and only if $ux \in U$ " (see [24, Lemma 2.4.]).

The set $B = \{ uru^{-1} \mid u \in U, r \in R \}$ is a subset of $\bar{R}$ ($\bar{R} \subseteq K$), so every element of B can be written as a word in the elements of A (uniquely), as in the proof above . If we denote the resulting set by B_1, passing from B to B_1 is known as the Reidemeister- Schreier rewriting process. With the above presentation $< A \mid B_1 >$ is a presentation for H. For detailed proof see Johnson [24, section 2] or Magnus, Karass and Solitar [28, Theorem 2.9].

1.3.3 Computer Programs:

In this section we give some details of the computer programs used in our work.

A Todd - Coxeter Program (TC):

Over the past thirty years the Todd - Coxeter algorithm has been implemented on computers in a number of different ways. The difference between them is the way of defining the cosets. Briefly we are going to mention the methods:

(1) HLT method: the primary aim in defining the cosets is to close at least one row of some relation table in order to get at least one deduction as soon as possible. With this method it is usual that many redundant cosets occur.

(2) Felsch method: the vacant places in the coset tables are filled row by row and after each definition all relations are scanned. With this method fewer redundant definitions are usually made but scanning after each definition is time consuming. In TC algorithm we filled the relation tables. Also it is possible to fill the coset table as well.

(3) Lookahead method: this method to some extent is a combination of the previous two methods. It involves using HLT for a specified period, then follows intensive scanning with no definitions being made.

At the University of St. Andrews we have a modern implementation of the Todd - Coxeter algorithm. In this implementation cosets may be enumerated by the Lookahead or Felch method. The TC program also provides for the manipulation of partial and complete coset tables. This version also contains an implementation of the Reidemeister - Schreier algorithm, added by E.F.Robertson, which enables the user to obtain defining relations for a subgroup H of finite index. In Chapters 4 , 5 and 6 we have used the TC program to obtain generators for a maximal subgroup of PSL(2,p). In order to do this we have used in the TC program the option RC.

RC: This finds nontrivial subgroups of a group i with index a multiple of a desired index. First construct a partial coset table with more rows than the desired index. Then repeatedly put cosets coincident with coset 1 and observe what happens. The first specified coset to be put coincident with coset 1 is read in and then the next coset is used if a favourable result does not occur. This process is repeated until a favourable result occurs or else we reach coset 1 or coset stop. In Chapter 7 we have used the TC program in order to find the index of a given subgroup.

CAYLEY

CAYLEY is a high level programming language designed to support convenient and efficient computation with groups and with other algebraic structures that arise naturally in the study of groups[15].

CAYLEY is designed so as to allow interactive execution of single statements, thus giving the user a high degree of control over the path taken in a computation. It is also possible to submit batch jobs, running through a specified list of commands.

CAYLEY has built into it a large amount of elementary group theory. We can compute with groups of the following types:

(1) A finitely presented group.

(2) A group of permutations acting on a finite set.

(3) A group of matrices over $\mathbb{Z}$.

(4) A group of matrices over GF(q).

We can carry out the following calculations:

(1) Calculations with group elements.

(2) Calculation with sets of group elements.

(3) Calculations with subgroups and quotient groups of a group.

(4) The construction of homomorphisms between two groups.

(5) Calculation of the order.

(6) Determination of conjugacy classes, normal subgroups, and the automorphism group of a finite group.

Standard Functions:

Over the years a considerable number of powerful algorithms have been developed for computing structural information about groups. The library associated with CAYLEY contains efficient implementations of these algorithms, which are available to the user in the form of Standard functions. The contents of the library of standard functions in CAYLEY is outlined in [15].

In connection with our results in Chapters 4, 5 and 6 we shall use some of the standard functions.

CHAPTER 2

ON F(n) GROUPS

2.0 INTRODUCTION:

In the first three sections in this chapter we study the groups with the presentation $F(n) = \langle R,S \mid R^n = S^n = (R^{a_1}S^{b_1})^{x_1}(R^{c_1}S^{d_1})^{y_1}(R^{a_2}S^{b_2})^{x_2}(R^{c_2}S^{d_2})^{y_2} \ldots (R^{a_m}S^{b_m})^{x_m}(R^{c_m}S^{d_m})^{y_m} = 1 \rangle$. Our actual aim is to investigate the structure of these groups. For some values of n, a_i, b_i, c_i, d_i, x_i, y_i we give some results on those groups where we have been able to determine their order, either finite or infinite. In the last section in this chapter we study the groups generated by A and B and subject to the following relations:

$A^4 = 1$,

$B^4 = 1$,

$(B(AB)^2)^4 = 1$,

$(B(BA)^6)^4 = 1$,

$(B(BA)^{14})^4 = 1$

"

"

"

$(B(BA)^{(2^{(n-1)/2}-2)})^4 = 1,$

$(A^{-1}B^{-1})^{2^{(n-3)/2}}B(BA)^{(2^{(n-1)/2}-2)}B(BA)^{2^{(n-3)/2}}B^{-1}(A^{-1}B^{-1})^{(2^{(n-1)/2}-2)}B^{-1}$

$(A^{-1}B^{-1})^{(2^{(n+1)/2}-3)}A(BA)^{(2^{(n-1)/2}-1)}B^{-1} = 1,$

$(BA)^{2^{(n-1)/2}}B^{-1}(A^{-1}B^{-1})^{(2^{(n-1)/2}-2)}B^{-1}(A^{-1}B^{-1})^{(2^{(n+1)/2}-3)}A^2 = 1.$

For $n \geq 5$, n is an odd integer, we have been able to determine the structure of these groups. We conclude by looking at the groups generated by A, B and subject to the following relations:

$A^4 = 1,$

$B^4 = 1,$

$(B(AB)^2)^4 = 1,$

$(B(BA)^6)^4 = 1,$

$(B(BA)^{14})^4 = 1,$

"

"

"

$(B(BA)^{(2^{n/2}-2)})^4 = 1,$

$B^{-1}(BA)^{2^{(n-2)/2}}B(BA)^{(2^{n/2}-2)}B(A^{-1}B^{-1})^{(2^{(n-2)/2}-1)}A(BA)^{(2^{n/2}-1)} = 1,$

$(BA)^{(2^{n/2}+2^{(n-2)/2}+2)}B(BA)^{(2^{n/2}-2)}B(A^{-1}B^{-1})^{(2^{(n-2)/2}-1)}A^2 = 1.$

For $n \geq 4$, n is an even integer, we have been able to determine the structure of these groups.

2.1 RESULTS ON F(n) GROUPS:

In this section we show how the orders of some of the groups F(n) may be determined. We conclude by looking at the special cases of the groups F(n) .

We denote $F(n) = \langle R,S \mid R^n = S^n = (R^{a_1}S^{b_1})^{x_1}(R^{c_1}S^{d_1})^{y_1}(R^{a_2}S^{b_2})^{x_2}(R^{c_2}S^{d_2})^{y_2}$ $\ldots (R^{a_m}S^{b_m})^{x_m}(R^{c_m}S^{d_m})^{y_m} = 1 \rangle$ .... (1).

Because $R^n = S^n = 1$ we can suppose that a_i, b_i, c_i, d_i, $1 \le i \le m$ are reduced modulo n.

Theorem 2.1.1 Let $F(n)$ be as in (1) and $a_i = b_i = 1$, $c_i = d_i = -1$, and $(x_1,x_2,x_3, \ldots, x_m,y_1,y_2, \ldots ,y_m) = t$. Then

(i) If $n \ge 4$ and if $t \ge 2$ then $F(n)$ is infinite.

(ii) If $n = 3$ and if $t \ge 3$ then $F(n)$ is infinite.

Proof : (i) Since $(x_1,x_2,x_3, \ldots, x_m,y_1,y_2, \ldots ,y_m) = t$ we may add the relation $(RS)^t = 1$ to get a homomorphic image of $F(n)$ with presentation $\langle R,S \mid R^n = S^n = (RS)^t = 1 \rangle$ which is infinite if $n \ge 4$ and $t \ge 2$, by [17].

(ii) Since $(x_1,x_2,x_3, \ldots, x_m,y_1,y_2, \ldots ,y_m) = t$ we may add the relation $(RS)^t = 1$ to get a homomorphic image of $F(n)$ with presentation $\langle R,S \mid R^3 = S^3 = (RS)^t = 1 \rangle$ which is infinite if $t \ge 3$ by [17].

Theorem 2.1.2 Let $F(n)$ be as in (1) and $x_i = z_i$, $a_i = b_i = 1$, $c_i = d_i = -1$ in $F(n)$. Let $Y_i = R^iS^i$, $1 \le i \le (n-1)$ and let $H(n)$ to be the subgroup of $F(n)$ generated by $\{ Y_i \mid 1 \le i \le (n-1) \}$. Then for every $n > 0$, $z_i, y_i, a_i, b_i, c_i, d_i \in \mathbb{Z}$, $H(n)$ is a normal subgroup with index n in $F(n)$ and has the presentation

$$\langle X_2, X_3, \ldots, X_n \mid \quad \prod_{i=1}^{m} ((X_2^{-1}X_3^{-1} \cdots X_{n-1}^{-1}X_n^{-1})^{z_i}(X_n)^{-y_i}) = 1,$$

$$\prod_{i=1}^{m} ((X_2)^{z_i}(X_n^{-1}X_{n-1}^{-1} \cdots X_3^{-1}X_2^{-1})^{y_i}) = 1,$$

$$\prod_{i=1}^{m} ((X_3)^{z_i}(X_2)^{-y_i}) = 1,$$

$$\prod_{i=1}^{m} ((X_4)^{z_i}(X_3)^{-y_i}) = 1,$$

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$$\prod_{i=1}^{m} ((X_n)^{z_i} (X_{n-1})^{-y_i}) = 1 \quad \rangle.$$

Proof: We can define cosets 1,2,3, ..., n of H(n) in F(n) by H(n) = 1 and iR = i + 1. No collapses occur. Now using the R.S. algorithm we can find a presentation for H(n). We can take $U = \{ R^i \mid i=0, \dots, n-1 \}$ as a Schreier transversal for H(n) in F(n). Then $\{ pq\underline{pq}^{-1} \mid p \in U, q \in \{R,S\} \}$ generates H(n) whenever pq is an element of U and $pq\underline{pq}^{-1}$ belongs to H(n) and $\{ prp^{-1} \mid p \in U, r \in \{R^n, S^n, \prod_{i=1}^{m} ((RS)^{z_i} (R^{-1}S^{-1})^{y_i}) \} \}$ is the set of relations. So we will get

<u>Generators of H(n)</u>

$X_i = R^{i+1}SR^{-i}$, $i = 1,2, \dots, n-2$

$X_{n-1} = SR^{-n+1}$

$X_n = RS$

$X_{n+1} = R^n$

<u>Relations of H(n)</u>

$D_{1+i} = R^iS^nR^{-i}$, $i = 0, 1,2, \dots, n-1$

$D_{n+1+i} = R^i (\prod_{k=1}^{m} ((RS)^{z_k} (R^{-1}S^{-1})^{y_k}) R^{-i} \quad k = 1,2, \dots, m \ ; i = 0,1, \dots, n$

Consequently,

$H(n) = \langle X_1, X_2, \dots, X_{n+1} \mid D_i = 1, \ i = 1, 2, \dots, 2n+1 \quad \rangle.$

We have to describe D_i in terms of X_i. We see that $X_{n+1} = 1$ and $D_{2n+1} = 1$, that is, H(n) can be generated by $X_1, X_2, \dots, X_n$. Consider the foregoing relations. Then we will get

$D_1 = (X_{n-1}, X_{n-2}, \dots, X_1, X_n) = 1$

$D_2 = (X_n, X_{n-1}, ..., X_2, X_1) = 1$

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$D_n = (X_{n-2}, X_{n-3}, ..., X_n, X_{n-1}) = 1$

$$D_{n+1} = \prod_{i=1}^{m} ((X_n)^{z_i} (X_{n-1})^{-y_i}) \qquad , \ i = 1,2, ... , m$$

$$D_{n+2} = \prod_{i=1}^{m} ((X_1)^{z_i} (X_n)^{-y_i}) \qquad , \ i = 1,2, ... , m$$

$$D_{n+3} = \prod_{i=1}^{m} ((X_2)^{z_i} (X_1)^{-y_i}) \qquad , \ i = 1,2, ... , m$$

$$D_{n+4} = \prod_{i=1}^{m} ((X_3)^{z_i} (X_2)^{-y_i}) \qquad , \ i = 1,2, ... , m$$

$$D_{n+5} = \prod_{i=1}^{m} ((X_4)^{z_i} (X_3)^{-y_i}) \qquad , \ i = 1,2, ... , m$$

"

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$$D_{2n} = \prod_{i=1}^{m} ((X_{n-1})^{z_i} (X_{n-2})^{-y_i}) \qquad , \ i = 1,2, ... , m$$

So for every i, $i = 2,3, ... , n$ the relations $D_i = 1$ are consequences of relation D_1. Eliminating $X_1 = X_2^{-1}X_3^{-1} ... X_{n-1}^{-1}X_n^{-1}$ we will get the required presentation for H(n).

Theorem 2.1.3 Let H(n) be as in Theorem 2.1.2 and let F(n) be as in (1) and $a_i = b_i = 1$, $c_i = d_i = -1$ in (1). If

* .... $\{ (z_i \in \mathbb{Z}, z_1 = z_2 = ... = z_{i-1} = z_{i+1} = ... = z_m = 0)$ and

$(y_j = 1, y_1 = ... = y_{j-1} = y_{j+1} = ... = y_m = 0) \}$ or

**........... $\{ (z_k = 1, z_1 = z_2 = ... = z_{k-1} = z_{k+1} = ... = z_m = 0)$ and

$(y_t \in \mathbb{Z}, y_1 = y_2 = ... = y_{t-1} = y_{t+1} = ... = y_m = 0) \}$ then

(i) H(n) is infinite; (when n is even and $z_i = -1$ in *) or

(when n is even and $y_t = -1$ in **)

(ii) Otherwise, H(n) is a cyclic group of order, $| z_i^{n-1} + z_i^{n-2} + \dots + z_i^2 + z_i + 1 |$ or $| y_j^{n-1} + y_j^{n-2} + \dots + y_j^2 + y_j + 1 |$

(iii) F(n) is a metacyclic group of order $n.| z_i^{n-1} + z_i^{n-2} + \dots + z_i^2 + z_i + 1 |$ or $n.| y_j^{n-1} + y_j^{n-2} + \dots + y_j^2 + y_j + 1 |$ if n is prime.

Proof: (i) Without loss of generality we can assume that the conditions in * have been met. Then from Theorem 2.1.2 the presentation for H(n) will become

$$\begin{aligned} \langle X_2, X_3, \dots, X_n \mid \quad & (X_2^{-1}X_3^{-1} \dots X_{n-1}^{-1}X_n^{-1})^{z_i} (X_n)^{-1} = 1, \\ & (X_2)^{z_i} (X_n^{-1}X_{n-1}^{-1} \dots X_3^{-1}X_2^{-1}) = 1, \\ & (X_3)^{z_i} (X_2)^{-1} = 1, \\ & (X_4)^{z_i} (X_3)^{-1} = 1 \\ & \quad " \\ & \quad " \\ & (X_n)^{z_i} (X_{n-1})^{-1} = 1 \ \rangle. \end{aligned}$$

As we can see $X_{n-1} = X_n^{z_i}$, $X_{n-2} = X_n^{z_i^2}$, $X_{n-3} = X_n^{z_i^3}$, ..., $X_2 = X_n^{z_i^{n-2}}$ and replacing these in the relations in H(n) we will get the following presentation for $H(n) = \langle X_n \mid X_n^q = 1, q = | z_i^{n-1} + z_i^{n-2} + \dots + z_i^2 + z_i + 1 | \rangle$. If n is even and z_i is -1 then $H(n) \cong \mathbb{Z}$ because -1 is a root of the $z_i^{n-1} + z_i^{n-2} + \dots + z_i^2 + z_i + 1$. A similar proof can be given if the conditions in ** have been satisfied.

(ii) Assume the conditions in * have been satisfied and then by (i) H(n) has the presentation $\langle X_n \mid X_n^q = 1 \rangle$, where $q = | z_i^{n-1} + z_i^{n-2} + \dots + z_i^2 + z_i + 1 |$. If n is not even and z_i is not -1 then H(n) is a cyclic group of order $| z_i^{n-1} + z_i^{n-2} + \dots + z_i^2 + z_i + 1 |$. A similar proof can be given if the conditions in ** have been satisfied.

(iii) Assume the conditions in * have been met. If n is not even and z_i is not -1 then by (ii) H(n) is the cyclic group of order $| z_i^{n-1} + z_i^{n-2} + ... + z_i^2 + z_i + 1 |$. But by Theorem 2.1.1 the index of H(n) in F(n) is n and H(n) is a normal subgroup, so F(n) is a metabelian group of order $n.| z_i^{n-1} + z_i^{n-2} + ... + z_i^2 + z_i + 1 |$. In particular if n >2, n prime, then by (ii) H(n) is a cyclic group of order $| z_i^{n-1} + z_i^{n-2} + ... + z_i^2 + z_i + 1 |$. But by Theorem 2.1.1 the index of H(n) in F(n) is n and H(n) is a normal subgroup. On the other hand n is prime, so F(n) is a metacylic group of order $n.| z_i^{n-1} + z_i^{n-2} + ... + z_i^2 + z_i + 1 |$. A similar proof can be given if the conditions in ** are satisfied. In (1) if we take $z_1 = a$, $y_1 = b$ and $z_i = 0 = y_i$, $i = 2,3, \dots, m$ then the presentation for F(n) will become as follows

(2)..... $F(n) = \langle R,S \mid R^n = S^n = (RS)^a(R^{-1}S^{-1})^b = 1 \rangle$.

If n = 3 then the presentation for F(3) will become as follows

(3)... $F(3) = \langle R,S \mid R^3 = S^3 = (RS)^a(R^{-1}S^{-1})^b = 1 \rangle$. Since $R^3 = 1 \Rightarrow R^{-1} = R^2$ and since $S^3 = 1 \Rightarrow S^{-1} = S^2$ the presentation for F(3) can be written as

(4)... $F(3) = \langle R,S \mid R^3 = S^3 = (RS)^a(R^2S^2)^b = 1 \rangle$

Let $K = \langle RS, R^2S^2 \rangle$ be the subgroup of F(3).

Theorem 2.1.4 K is a normal subgroup of index 3 in F(3) and has the presentation

$$\langle A,E \mid A^a(EA)^b = 1,\ E^aA^{-b} = 1,\ (A^{-1}E^{-1})^aE^{-b} = 1 \rangle$$

Proof: We can define cosets 1,2,3 of K in F(3) as K =1, 1R = 2 and 2R = 3. No collapses occur. Now using the R.S. algorithm we can find a presentation for K. We can take $U = \{ R^i \mid i = 0, 1, 2 \}$ as a Schreier transversal for K in F(3). Then $\{ ux\underline{ux}^{-1} \mid u \in U, x \in \{R, S\} \}$ generates K whenever $\underline{ux}$ is an element of U and $ux\underline{ux}^{-1}$ belongs to K and $\{ uru^{-1} \mid u \in U,\ r \in \{ R^3, S^3, (RS)^a(R^{-1}S^{-1})^b \}\}$ is the set of relations . So we will get

<u>Generators of K:</u> $A = RS$, $B = R^3$, $C = SR^{-2}$, $E = R^3SR^{-1}$.

<u>Relations of K :</u> written in terms of A, B, C, E ;

$R^3 = B$

$S^3 = CEA$

$RS^3R^{-1} = ACE$

$R^2S^3R^{-2} = EAC$

$(RS)^a(R^2S^2)^b = A^a(EA)^b$

$R(RS)^a(R^2S^2)^bR^{-1} = E^a(BCE)^b$

$R^2(RS)^a(R^2S^2)^bR^{-2} = B(CB)^aB(AC)^b$

Since B = 1 and after eliminating C we get the presentation for K as claimed.

Corollary 2.1 If $(a,b) = t$ and $t \geq 3$ then the group F(3) is infinite.

Proof: In Theorem 2.1.1 if we take $x_1 = a$, $y_1 = b$, $x_i = 0 = y_i$, $i = 2,3,4, \dots, m$ then if $(a,b) = t \geq 3$ by Theorem 2.1.1 (ii) F(3) is an infinite group.

Corollary 2.2 Let F(3) be as in (4) if $b = 1$ then F(3) is a metacyclic group of order $3(a^2 + a + 1)$.

Proof: By Theorem 2.1.4 K is a normal subgroup of F(3) and has the presentation $\langle A,E \mid A^a(EA)^b = 1,\ E^aA^{-b} = 1,\ (A^{-1}E^{-1})^aE^{-b} = 1 \rangle$. If $b = 1$ then the presentation for K will become $\langle E \mid E^{a^2+a+1} = 1 \rangle$ so K is a cyclic subgroup of order $(a^2 + a + 1)$. Since by Theorem 2.1.4 K is normal with index 3 so F(3)/K is cyclic. Therefore F(n) is a metacyclic group of order $3(a^2 + a + 1)$.

Lemma 2.1 Let F(3) be as in (4) and K as in Theorem 2.1.4. Suppose $(a,b) = 2$ then

(i) A^2 or E^2 is central in K.

(ii) If $(a^2,b) = 2$ and $(a,b^2) = 2$ then A^2 and E^2 are central in K.

Proof (i): From relation 1 in K we have $A^a = (EA)^{-b}$. Taking the a th power of both sides of this equation we get $A^{a^2} = (EA)^{-ba}$ ...... (*). From relation 3 in K we have $E^b = (EA)^{-a}$. Taking both sides of this equation to the b th power we get $E^{b^2} = (EA)^{-ab}$ ..... (**). From (*) and (**) we obtain $A^{a^2} = E^{b^2}$. If $(a,b) = 2 \Rightarrow$ $(a^2,b) = 2$ or $(a,b^2) = 2$. If $(a^2,b) = 2 \Rightarrow \exists$ f, g $\in \mathbb{Z}$ such that $2 = fa^2 + gb$ and therefore

$$A^2 = A^{fa^2}.A^{gb}$$

$$A^2 = (E^{b^2})^f.(E^a)^g \qquad \text{since } A^{a^2} = E^{b^2} \text{ and } E^a = A^b$$

$$A^2 = E^{fb^2 + ga}$$

$\Rightarrow A^2$ is central in K.

If $(a,b^2) = 2 \Rightarrow \exists$ q, w $\in \mathbb{Z}$ such that $2 = qa + wb^2$ and therefore

$$E^2 = E^{qa}.E^{wb^2} \qquad \text{since } A^{a^2} = E^{b^2} \text{ and } E^a = A^b$$

$$E^2 = A^{qb + wa^2}$$

$\Rightarrow E^2$ is central in K.

(ii) The proof is trivial from (i).

Lemma 2.2 Let K be as in Theorem 2.1.4 and $(a,b) = 2$.

(i) If $(a^2,b) = 2$ and $(a,b^2) = 2$ then the subgroup $D = \langle A^2, E^2 \rangle$ of K has index 4 in K and has the presentation

$$\langle J, L, M, N, P \mid J^{a/2}(MPJ)^{b/2} = 1, L^{a/2}J^{-b/2} = 1, L^{b/2}(MPJ)^{a/2} = 1, J^{a/2}(NL)^{b/2} = 1,$$
$$P^{a/2}J^{-b/2} = 1, P^{b/2}(NL)^{a/2} = 1, (MN)^{a/2}(LN)^{b/2} = 1,$$
$$L^{a/2}(N^{-1}M^{-1})^{b/2} = 1, L^{b/2}(LN)^{a/2} = 1, (NM)^{a/2}PJ(MPJ)^{(b-2)/2}M = 1,$$
$$P^{a/2}(M^{-1}N^{-1})^{b/2} = 1, P^{b/2}PJ(MPJ)^{(a-2)/2}M = 1 \rangle$$

where $J = A^2$, $L = E^2$, $M = EAE^{-1}A^{-1}$, $N = AEAE^{-1}$, $P = AE^2A^{-1}$.

(ii) If $(a^2,b) = 2$ and $(a,b^2) = 2$ then $D = \langle A^2, E^2 \rangle$, the subgroup of K, is a cyclic group of order $2(q^2 + wq + w^2)$, where $q = a/2$, $w = b/2$.

Proof (i): If $(a^2,b) = 2$ and $(a,b^2) = 2$ then by Lemma 2.1 (ii) A^2 and E^2 are central in K. So the subgroup D is normal in K. Now we can consider the factor group

$K/D = \langle A,E \mid A^a(EA)^b = 1,\ E^aA^{-b} = 1,\ (A^{-1}E^{-1})^aE^{-b} = 1, A^2 = 1, E^2 = 1 \rangle$

$\quad = \langle A,E \mid A^2 = 1, E^2 = 1,\ (EA)^b = 1,\ (EA)^a = 1 \rangle.$

Since $(a,b) = 2$ so $\exists$ f, g $\in \mathbb{Z}$ such that $2 = fa + gb$ and therefore $(EA)^2 = (EA)^{fa}.(EA)^{gb}$. Since $(EA)^a = 1$ and $(EA)^b = 1$, therefore $(EA)^2 = 1$, because $A^2 = 1$ and $E^2 = 1$ we get that K/D is an abelian group of order 4. This means that the index of D in K is 4. We can take $U = \{ 1, A, E, AE \}$ as a Schreier transversal for D in K. Then $\{ ux\underline{ux}^{-1} \mid u\in U, x\in \{A, E\} \}$ generates D whenever $\underline{ux}$ is an element of U and $ux\underline{ux}^{-1}$ belongs to D and $\{uru^{-1} \mid u\in U, r \in \{ A^a(EA)^b,\ E^aA^{-b},\ (A^{-1}E^{-1})^aE^{-b} \}\}$ is the set of relations . So we will get

<u>Generators of D:</u> $J = A^2, L = E^2,\ M = EAE^{-1}A^{-1}, N = AEAE^{-1}, P = AE^2A^{-1}$.

<u>Relations of D :</u> written in terms of J, L, M, N, P

(I) $A^a(EA)^b = J^{a/2}(MPJ)^{b/2}$

(II) $E^aA^{-b} = L^{a/2}J^{-b/2}$

(III) $E^b(EA)^a = L^{b/2}(MPJ)^{a/2}$

(IV) $A^{a+1}(EA)^bA^{-1} = J^{a/2}(NL)^{b/2}$

(V) $AE^aA^{-b-1} = P^{a/2}J^{-b/2}$

(VI) $AE^b(EA)^aA^{-1} = P^{b/2}(NL)^{a/2}$

(VII) $AE^a(EA)^bE^{-1} = (MN)^{a/2}(LN)^{b/2}$

(VIII) $E^{a+1}A^{-b}E^{-1} = L^{a/2}(N^{-1}M^{-1})^{b/2}$

(IX) $E^{b+1}(EA)^aE^{-1} = L^{b/2}(LN)^{a/2}$

(X) $AEA^a(EA)^b = (NM)^{a/2}PJ(MPJ)^{(b-2)/2}M$

(XI) $AEE^aA^{-b}(AE)^{-1} = P^{a/2}(M^{-1}N^{-1})^{b/2}$

(XII) $AEE^b(EA)^a(AE)^{-1} = P^{b/2}PJ(MPJ)^{(a-2)/2}M$

We get the presentation for D as claimed.

Proof (ii): Let $(a,b) = 2$. Suppose $(a^2,b) = 2$ and $(a,b^2) = 2$. Then from Lemma 2.1 A^2 and E^2 are central in K. So $D = \langle A^2, E^2 \rangle$ is an abelian group. Now using the commutativity of D, the relations of D in **(i)** can be written as

(1)	$J^{a/2+b/2}M^{b/2}P^{b/2} = 1,$	(7)	$L^{b/2}M^{a/2}N^{a/2+b/2} = 1,$
(2)	$J^{-b/2}L^{a/2} = 1,$	(8)	$L^{a/2}M^{-b/2}N^{-b/2} = 1,$
(3)	$J^{a/2}L^{b/2}M^{a/2}P^{a/2} = 1,$	(9)	$L^{b/2+a/2}N^{a/2} = 1,$
(4)	$J^{a/2}L^{b/2}N^{b/2} = 1,$	(10)	$J^{b/2}M^{a/2+b/2}N^{a/2}P^{b/2} = 1,$
(5)	$J^{-b/2}P^{a/2} = 1,$	(11)	$M^{-b/2}N^{-b/2}P^{a/2} = 1,$
(6)	$L^{a/2}N^{a/2}P^{b/2} = 1,$	(12)	$J^{a/2}M^{a/2}P^{a/2+b/2} = 1$

Now we are going to show that the relations (6), (9), (10), (11) are redundant.

From relation (2) and (5) we derive $J^{b/2} = L^{a/2} = P^{a/2}$ ....(i).

Consider relation (6) $L^{a/2}N^{a/2}P^{b/2} = 1$

$N^{a/2} = P^{-b/2}L^{-a/2}$.

Using (i) we get $N^{a/2} = P^{-a/2-b/2}$ ....(ii).

Consider relation (10) $J^{b/2}M^{a/2+b/2}N^{a/2}P^{b/2} = 1,$

$M^{a/2+b/2}N^{a/2}P^{a/2+b/2} = 1$ using (i)

$M^{a/2+b/2} = 1.$ using (ii)

So relation (10) can be replaced by $M^{a/2+b/2} = 1$ ....(10*).

Consider relation (11) $M^{-b/2}N^{-b/2}P^{a/2} = 1$

$M^{-b/2}N^{-b/2}L^{a/2} = 1$ using (i)

So relation (11) is redundant because of relation (8).

Consider relation (6) $L^{a/2}N^{a/2}P^{b/2} = 1$

$N^{a/2}P^{b/2+a/2} = 1$ using (i)

So the relation (6) can be replaced by $N^{a/2}P^{b/2+a/2} = 1$ ....(6*).

From relation (3) and (12) we derive $L^{b/2} = P^{b/2}$ .... (iii).

Consider (6*) $N^{a/2}P^{b/2+a/2} = 1$ and using (iii) we get $N^{a/2}L^{a/2+b/2} = 1$ but this is redundant because of relation (9) so (6*) is redundant and consequently the relation (6) is redundant.

From relations (1) and (12) and using (i) we derive $M^{b/2} = M^{a/2}$ .... (iv).

Consider relation (8) $L^{a/2}M^{-b/2}N^{-b/2} = 1$ we can rewrite this relation as

$$N^{b/2} = L^{a/2}M^{-b/2}. \qquad \text{...................(v)}$$

Consider relation (7) $L^{b/2}M^{a/2}N^{a/2+b/2} = 1$

$$L^{b/2+a/2}N^{a/2} = 1 \quad \text{using (iv)}$$

So the relation (9) is redundant.

The relation (4) can be rewritten as $N^{-b/2} = J^{a/2}L^{b/2}$ .... (vi).

Consider relation (8) $L^{a/2}M^{-b/2}N^{-b/2} = 1$ using (vi) we get $L^{a/2+b/2}J^{a/2}M^{-b/2} = 1$...(vii).

Consider relation (3) $J^{a/2}L^{b/2}M^{a/2}P^{a/2} = 1$ using (i) we get $J^{a/2}L^{a/2+b/2}M^{a/2} =1$...(viii).

From (vii) and (viii) we derive $M^{-b/2} = M^{a/2}$ so the relation (10*) is redundant. Consequently relation (10) is also redundant. After all the computations the remaining relations are (1), (2), (3), (4), (5), (7), (8), (12).

Now considering $(a,b) = 2 \Rightarrow \exists$ q, w $\in \mathbb{Z}$ such that $a = 2q$ and $b = 2w$ and $(q,w) = 1$. Considering the fact that $(a^2,b) = 2$ and $(b^2,a) = 2$ so w and q are both odd integers. We can substitute respectively 2q and 2w instead of a and b in the relations of D. On the other hand D is an abelian group and so its order is given by the invariant factors of the relation matrix M

$$\begin{bmatrix} q+w & 0 & w & 0 & w \\ -w & q & 0 & 0 & 0 \\ q & w & q & 0 & q \\ q & w & 0 & w & 0 \\ -w & 0 & 0 & 0 & q \\ 0 & w & q & q+w & 0 \\ 0 & q & -w & -w & 0 \\ q & 0 & q & 0 & q+w \end{bmatrix}$$

$2(q^2 + qw + w^2)$ is the only invariant factor of M, h1(M) has to be 1 because h.c.f.$\{q,w,q+w,0\}=1$,

h2(M) has to 1 because h.c.f$\{\det(\begin{bmatrix} q & -w \\ w & 0 \end{bmatrix}) = w^2, \det(\begin{bmatrix} 0 & q \\ q & q \end{bmatrix}) = q^2\} = 1$,

h3(M) has to be 1 because

$$\text{h.c.f.}\{\det(\begin{bmatrix} -w & 0 & 0 \\ 0 & q & -w \\ q & w & 0 \end{bmatrix}) = w^3, \det(\begin{bmatrix} q & 0 & 0 \\ 0 & 0 & q \\ 0 & q & q+w \end{bmatrix}) = q^3\} = 1,$$

h4(M) has to be 1 because

$$\text{h.c.f}\{\det(\begin{bmatrix} -w & q & 0 & 0 \\ -w & 0 & 0 & 0 \\ 0 & q & -w & -w \\ q & w & 0 & q \end{bmatrix}) = w^3 q,$$

$$\det(\begin{bmatrix} -w & 0 & 0 & q \\ 0 & q & -w & 0 \\ q & 0 & 0 & q+w \\ 0 & w & q+w & 0 \end{bmatrix}) = (wq+w^2+q^2)^2\} = 1. \text{ Let } p = (q^2+wq+w^2).$$

Considering the fact that q and w are odd integers. h5(M) is $2(q^2+wq+w^2)$ because $h5(M) = \text{h.c.f}\{ wq(w+q).p,\ w(w+q)(w-q)p,\ w^2(w+q).p,\ wq(w-q)p,\ 2qw^2p,\ w(w^2+q^2)p,\ w^2(w-q)p,\ 2q^2wp,\ q^2(w+q)p,\ (w-q)p^2,\ (w+q)(w^2-wq+q^2)p,\ q(w-q)(w+q)p\} = 2p$. So $2(q^2+wq+w^2)$ is the only invariant factor of M. Therefore D is a cyclic group of order $2(q^2 + wq + w^2)$.

Theorem 2.1.5 Let $(a,b) = 2$, $(a^2,b) = 2$, $(a,b^2) = 2$ and let F(3) be as in (4). Then F(3) has order $24(q^2 + qw + w^2)$, where $q = a/2$, $w = b/2$.

Proof: By Theorem 2.1.4 F(3) has $K = \langle RS, R^2S^2 \rangle$ as a normal subgroup of index 3 and by Lemma 2.2 **(i)**, K has $D = \langle A^2, E^2 \rangle$ as a cyclic subgroup of index 4 and by Lemma 2.2 **(ii)**, the order of D is $2(q^2 + wq + w^2)$. Obviously the order of K is $8(q^2 + w^2 + wq)$ and the order of F(3) is $24(q^2 + wq + w^2)$.

Lemma 2.3 Let $(a,b) = 2$, $(a^2,b) = 2$ and let K be the group as in Theorem 2.1.4 . Then $H = \langle E, A^2 \rangle$ is a normal subgroup of K with index 2 and the order of H is $4(q^2 + wq + w^2)$. (Here $q = a/2$, $w = b/2$.)

Proof: Let $(a,b) = 2$, $(a^2,b) = 2$. By the proof of Lemma 2.1 A^2 is central in K. Consider

$H = \langle E, A^2 \rangle < K$. Since A^2 is central in K, $EA^2 = A^2E$ so H is an abelian group.

Now we can consider the factor group

$K/H = \langle A,E \mid A^a(EA)^b = 1,\ E^aA^{-b} = 1,\ (A^{-1}E^{-1})^aE^{-b} = 1, A^2 = 1, E = 1 \rangle$. Since $(a,b) = 2$ so $K/H = \mathbb{Z}_2$. This means K/H has order 2 therefore the index of H in K is 2. We can take $U = \{ 1, A\}$ as a Schreier transversal for H in K. Then $\{ ux\underline{ux}^{-1} \mid u\in U, x\in \{A^2, E \} \}$ generates H whenever $\underline{ux}$ is an element of U and $ux\underline{ux}^{-1}$ belongs to H and $\{ uru^{-1} \mid u\in U,\ r \in \{ A^a(EA)^b,\ E^aA^{-b},\ (A^{-1}E^{-1})^aE^{-b} \}\}$ is the set of relations . So we will get

<u>Generators of H:</u> $V = E$, $Y = A^2$, $Z = AEA^{-1}$,

<u>Relations of H :</u> written in terms of V, Y, Z

(1) $A^a(EA)^b = V^{b/2}Y^{b/2+a/2}Z^{b/2}$

(2) $E^aA^{-b} = V^aY^{-b/2}$

(3) $E^b(EA)^a = V^{b+a/2}Y^{a/2}Z^{a/2}$

(4) $A^{a+1}(EA)^bA^{-1} = V^{b/2-1}Y^{a/2+b/2}Z^{1+b/2}$

(5) $AE^aA^{-b-1} = Y^{-b/2}Z^a$

(6) $AE^b(EA)^aA^{-1} = V^{a/2}Y^{a/2}Z^{a/2+b}$

Considering that H is an abelian group from relations (1) and (4) we can get

$$V = Z \quad \text{........................(i).}$$

Now substituting V instead of Z in the relations of H we see that

(4) is redundant because of relation (1)

(5) is redundant because of relation (2)

(6) is redundant because of relation (3).

With the remaining relations the presentation for H is as follows:

$\langle V, Y \mid V^bY^{b/2+a/2} = 1,\ V^aY^{-b/2} = 1,\ V^{b+a}Y^{a/2} = 1 \rangle$.

Now from relation 2 we derive $V^a = Y^{b/2}$ ....(ii).

Considering relation 1 and using (ii) we see that the relation 3 is redundant. After all the computations the presentation of H is as follows:

$$\langle V, Y \mid V^b Y^{b/2+a/2} = 1,\ V^a Y^{-b/2} = 1 \rangle.$$

Now since $(a,b) = 2 \Rightarrow \exists\ q, w \in \mathbb{Z}$ such that $a = 2q$ and $b = 2w$ and $(q,w) = 1$. Considering the fact that $(a^2,b) = 2$, w is an odd integer. We can substitute respectively 2q and 2w instead of a and b in the relations of H. On the other hand H is an abelian group and so its order is given by the invariant factors of the relation matrix M

$$M = \begin{bmatrix} 2w & w+q \\ 2q & -w \end{bmatrix}.$$

$2(q^2+wq+w^2)$ is the only invariant factor of M because

$h1(M) = \text{h.c.f}\{\ 2w, q+w, 2q, w\ \} = 1$ (since $(q,w) = 1$, w is odd so $(2q,w) = 1$),

$h2(M) = 2(w^2+q^2+qw)$. So $2(q^2+wq+w^2)$ is the only invariant factor of M. Therefore H is a cyclic group of order $2(q^2 + wq + w^2)$.

Theorem 2.1.6 Let $(a,b) = 2$, $(a^2,b) = 2$ and let F(3) be as in Theorem 2.1.4 then F(3) has order $24(q^2 + wq + w^2)$, where $q= a/2$, $w = b/2$.

Proof: By Theorem 2.1.4 F(3) has $K = \langle RS, R^2S^2 \rangle$ as a normal subgroup of index 3 and by Lemma 2.3 , K has $H = \langle A^2, E \rangle$ as a cyclic subgroup of index 2 and by Lemma 2.3 the order of H is $4(q^2 + wq + w^2)$. Obviously the order of K is $8(q^2 + w^2 + wq)$ and the order of F(3) is $24(q^2 + wq + w^2)$.

Theorem 2.1.7 Let $n \geq 3$ and let F(n) be as in (2) . If $(a,b) = 1$ then F(n) is a finite group of order $n.(a^n - b^n)/(a - b)$ and if n is a prime number then F(n) is a metacyclic group.

Proof: In Theorem 2.1.2 if we take $z_1 = a$, $y_1 = b$ and $z_i = 0 = y_i$, $i = 2,3, \dots , m$ then the presentation for H(n) will become:

$$\begin{aligned} \langle X_2, X_3, \dots, X_n \mid \quad & (X_n X_{n-1} \dots X_3 X_2)^a (X_n)^b = 1, \\ & (X_2)^a (X_n X_{n-1} \dots X_3 X_2)^b = 1, \\ & (X_3)^a (X_2)^{-b} = 1, \\ & (X_4)^a (X_3)^{-b} = 1 \\ & \quad '' \\ & \quad '' \\ & \quad '' \\ & (X_n)^a (X_{n-1})^{-b} = 1 \qquad \rangle \end{aligned}$$

From Theorem 2.1.2 H(n) is a normal subgroup of F(n) with index n.

From relations 1 and 2 we may obtain $X_n{}^{b^2} = X_2{}^{a^2}$ .... (F_o).

From relations (2+k) and (3+k) we may obtain $X_{1+k}{}^{b^2} = X_{3+k}{}^{a^2}$ .... (F_k).

Here $k = 1,2,3, \dots , (n-3)$.

On the other hand from relations (1+j) we may obtain $X_j^b = X_{j+1}^a$ ....R_j where $j = 2, \dots , (n-1)$.

Now we will show that X_l (where $l = 2,3, \dots , (n-1)$) can be written as power of X_n . Here we will give the proof when n is an odd integer.

Let $n-3 = i$ and $i \geq m$, $m = 0,1, \dots , i$. Because $(a,b) = 1$ so $(a^2,b) = 1$, therefore $fa^2 + gb = 1$.

Suppose i-m is even , then from (F_o) we have $X_n{}^{b^2} = X_2{}^{a^2}$. Raise both sides of this equation to the power $f^{\,i-m+1} a^{i-m} b^{i-m}$. If $(i-m) = 0$ use only (F_0). If $(i - m) \neq 0$ and $(i - m)$ is even, then using respectively $F_0, F_1, \dots, F_{(i-m)-1}$, we may obtain

$$(X_n{}^{b^2})^{f^{\,i-m+1} a^{i-m} b^{i-m}} = (X_{2+i-m}{}^{a^2})^{a^{2+2(i-m)} f^{\,i-m+1}} \quad . \text{.................................}(1)$$

Suppose i-m is odd, again raising both sides of the (F_o) to the power $f^{\,i-m+1} a^{i-m} b^{i-m}$ then using $F_1, F_2, F_3, \dots , F_{(i-m-1)}$ we may obtain

$$(X_n{}^{b^2})^{f^{i-m+1}a^{i-m}b^{i-m}} = (X_{i-m+1})^{a^{2(i-m)+1}f^{i-m+1}b} \quad \text{..............................(2)}$$

Now using the induction hypothesis we can show that X_{n-1-m} can be written as a power of X_n where $m = 0,1, \ldots, i$.

Let $m = 0$. Then $i - m$ is even. Using $m = 0$ and (1) and replacing $i = n-3$ we can get

$$(X_n{}^{b^2})^{f^{n-2}a^{n-3}b^{n-3}} = (X_{n-1})^{a^{2(n-2)}f^{n-2}}$$

$$= (X_{n-1})^{(b((fa^2)^{(n-2)}-1)/b+1} \quad \text{because } b|((fa^2)^{n-2}-1)$$

$$= (X_{n-1}{}^{b})^{((fa^2)^{(n-2)}-1)/b}X_{n-1} \text{ from } R_{n-1}\ (X_{n-1}{}^{b}) = (X_n{}^{a})$$

so $X_{n-1} = (X_n)^{(b^n f^{n-2}a^{n-3}-(a(fa^2)^{n-2}-a))/b}$.

Suppose for $m = t$ X_{n-1-t} has to be written as a power of X_n.

Now we will try to show that $X_{n-1-(t+1)}$ can be written as a power of X_n. Without loss of generality we can assume that t+1 is even. Raising both sides of F_o to the power

$$f^{i-(t+1)+1}a^{i-(t+1)}b^{i-(t+1)}$$

we can get $(X_n{}^{b^2})^{f^{i-(t+1)+1}a^{i-(t+1)}b^{i-(t+1)}} = (X_2{}^{a^2})^{f^{i-(t+1)+1}a^{i-(t+1)}b^{i-(t+1)}}$

Therefore by (1) the last equation can be written as

$$(X_n{}^{b^2})^{f^{i-(t+1)+1}a^{i-(t+1)}b^{i-(t+1)}} = (X_{2+i-(t+1)})^{a^{2+2(i-(t+1))}f^{i-(t+1)+1}}.$$

Replacing $i = n-3$ in the above equation we can get

$$(X_n{}^{b^2})^{f^{n-t-3}a^{n-t-4}b^{n-t-4}} = (X_{n-1-(t+1)})^{a^{2+2(n-3-(t+1))}f^{n-3-(t+1)+1}}$$

$$= (X_{n-1-(t+1)})^{a^{2(n-3-t)}f^{n-3-t}}$$

$$= (X_{n-1-(t+1)})^{((fa^2)^{(n-3-t)}-1)b/b+1} \quad \text{because } b|((fa^2)^{(n-3-t)}-1)$$

$$= (X_{n-1-(t+1)}{}^{b})^{((fa^2)^{(n-3-t)}-1)/b}X_{n-1-(t+1)}$$

$$= (X_{n-1-(t+1)}{}^{a})^{((fa^2)^{(n-3-t)}-1)/b}X_{n-1-(t+1)}$$

since from R_{n-1-t} $X^{b}_{n-1-(t+1)} = X^{a}_{n-1-(t+1)}$.

By the induction hypothesis X_{n-1-t} is a power of X_n as in the following

$$X_{n-1-t} = X_n{}^{(b^n f^{i+1-t}a^{i-t}-(b^n f^{i+1-(t-1)}a^{i-(t-1)}-(b^n f^{i+1-(t-2)}a^{i-(t-2)}-(\ldots}$$
$$-(b^n f^{i}a^{i-1}-(b^n f^{i+1}a^{i}-(a(fa^2)^{(i+1)}-a))(a(fa^2)^{(i+1)-1}-a))(a(fa^2)^{(i+1)-2}-a)).$$

$$(a(fa^2)^{(i+1)-3} - a))(a(fa^2)^{(i+1)-4} - a)) \dots (a(fa^2)^{(i+1)-(t-1)} - a))(a(fa^2)^{(i+1)-t} - a)).1/(b^{t+1})$$

Replacing X_{n-1-t} in the above equation we can get $X_{n-1-(t+1)}$ as a power of X_n as in the following

$$X_{n-1-(t+1)} = X_n^{(b^n f^{n-2-(t+1)} a^{n-3-(t+1)} -(b^n f^{n-2-t} a^{n-3-t} -(b^n f^{n-2-(t+1)} a^{n-3-(t-1)} -(\dots -(b^n f^{n-3} a^{n-4} -(b^n f^{n-2} a^{n-3} -(a(fa^2)^{(n-2)} -a))(a(fa^2)^{(n-2)-1} -a))(a(fa^2)^{(n-2)-2} - a)). (a(fa^2)^{(n-2)-3} - a))(a(fa^2)^{(n-2)-4} - a)) \dots (a(fa^2)^{(n-2)-(t-1)} - a))(a(fa^2)^{(n-2)-t} - a)).1/(b^{t+1})}$$

Now the induction is completed. For every m , m= 0,1, ... , i , X_{n-1-m} can be written as a power of X_n so this means H(n) is a cyclic group. To see this we can write X_{n-1-m} , m=0,1, ... , i as in the following

$$X_{n-1} = X_n^{(b^n f^{n-2} a^{n-3} -(a(fa^2)^{(n-2)} -a)).1/b}$$

$$X_{n-2} = X_n^{(b^n f^{n-3} a^{n-4} -(b^n f^{n-2} a^{n-3} -(a(fa^2)^{(n-2)} -a))(a(fa^2)^{(n-3)} -a)).1/(b^2)}$$

$$X_{n-3} = X_n^{(b^n f^{n-4} a^{n-5} -(b^n f^{n-3} a^{n-4} -(b^n f^{n-2} a^{n-3} -(a(fa^2)^{(n-2)} -a)). (a(fa^2)^{(n-3)} -a))(a(fa^2)^{(n-4)} -a)).1/(b^3)}$$

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"

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$$X_2 = X_n^{(b^n f -(b^n f^2 a -(b^n f^3 a^2 -(b^n f^4 a^3 -(\dots -(b^n f^{n-4} a^{n-5} -(b^n f^{n-3} a^{n-4} - (b^n f^{n-2} a^{n-3} -(a(fa^2)^{(n-2)} -a))(a(fa^2)^{(n-3)} -a))(a(fa^2)^{(n-4)} -a)) \dots). (a(fa^2)^4 -a))(a(fa^2)^3 -a))(a(fa^2)^2 -a))(a(fa^2) -a)).1/(b^{n-2})}$$

Replacing these in the relations of H(n) and after very long calculations the h.c.f of the powers of X_n can be found as $(a^n - b^n)/(a - b)$.

Now H(n) is a cyclic group and from Theorem 2.1.2 H(n) is a normal subgroup in F(n) and its index is n. When n is prime F(n)/H(n) is a cyclic group so F(n) is a metacyclic group. Obviously the order of F(n) is $n.(a^n - b^n)/(a - b)$, because by Theorem 2.1.2 the index of H(n) in F(n) is n and the order of H(n) is $(a^n - b^n)/(a - b)$.

The proof is similar when n is even.

2.2 RESULTS ON F(3) GROUPS:

The F(3) groups are the groups generated by the two elements R and S of order three subject to one more relation i.e. the groups

$$\langle R,S \mid R^3 = S^3 = w(R,S) = 1 \rangle$$

We give some results on those groups where we have been able to determine their order, either finite or infinite.

Theorem 2.2.1 Let $F = \langle R,S \mid R^3 = S^3 = (RS)^a(R^2S)^b(R^2S^2)^d = 1 \rangle$ and let $H = \langle R^2S, SR^{-1} \rangle$ be a subgroup of F then:

(i) If $a \equiv 1(\text{mod}3)$ and $d \equiv 1(\text{mod}3)$ the index of H in F is 3 and H has the following presentation

$$\begin{aligned}\langle A,B \mid\ & B^{-1}A^{-1}(BAB^{-1}A^{-1})^{(a-1)/3}(B^{-1}A^{-1})^bB^{-1}(ABA^{-1}B^{-1})^{(d-1)/3} = 1,\\ & A(B^{-1}A^{-1}BA)^{(a-1)/3}A^{b+1}B(A^{-1}B^{-1}AB)^{(d-1)/3} = 1,\\ & B(AB^{-1}A^{-1}B)^{(a-1)/3}B^bA^{-1}(B^{-1}ABA^{-1})^{(d-1)/3} = 1 \rangle,\end{aligned}$$

where $A=R^2S$, $B=SR^{-1}$.

(ii) If $a = 1 = d$, $b\in\mathbb{Z}$ then H is a cyclic group of order $b^2 + 3b + 3$. Obviously the order of F is $3(b^2 + 3b + 3)$ in this case.

(iii) If $b \equiv 1(\text{mod}3)$, $a \equiv 1(\text{mod}3)$, $d \equiv 1(\text{mod}3)$ then H is the derived subgroup of F.

Proof: (i) Let $H = \langle R^2S, SR^{-1} \rangle$ be a subgroup of F. We can define the cosets

$H = 1$, $1R = 2$, $2R = 3$. No collapses occur. We now consider coset representatives rather than cosets. Let us put $A = R^2S$, $B = SR^{-1}$. The coset table and coset representative table are respectively :

	R	S	R^{-1}	S^{-1}
1	2	2	3	3
2	3	3	1	1
3	1	1	2	2

	R	S	R^{-1}	S^{-1}
1	2	$B2$	3	$A^{-1}3$
2	3	$B^{-1}A^{-1}3$	1	$B^{-1}1$
3	1	$A1$	2	$AB2$

From the 3 rd relation of F we get the relations for H as claimed.

(ii) If $a = 1 = d$, $b \in \mathbb{Z}$ then by **(i)** the presentation for H will become $H = \langle A, B \mid (B^{-1}A^{-1})^{(b+1)}B^{-1} = 1, A^{b+2}B = 1, B^{b+1}A^{-1} = 1 \rangle$. From relation 3 we derive $A = B^{b+1}$ ......(*) so this means H is a cyclic group. Using (*) in the presentation of H we see that order of H is $b^2 + 3b + 3$. From **(i)** the index of H in F is 3. Therefore the order of F is $3(b^2 + 3b + 3)$ if $a = 1 = b$.

(iii) Abelianizing the relations of F and assuming that $b \equiv 1(\text{mod}3)$, $a \equiv 1(\text{mod}3)$, $d \equiv 1(\text{mod}3)$ then we can get the following presentation for the factor group $F/F' = \langle A,B \mid R^3 = S^3 = R^{3(t+2q+2w)+5}S^{3(t+q+2w)+4} = 1, RSR^{-1}S^{-1} = 1 \rangle$.

Using relations 1 and 2 we get $F/F' = \langle A, B \mid R^3 = S^3 = R^2S = 1, RSR^{-1}S^{-1} = 1 \rangle$. If we look at the generators of H we see that they are elements of F'. From F/F' we see that index of F' in F is 3 so the index of H is 3. Therefore $F' = H$.

Theorem 2.2.2 Let $G = \langle R,S \mid R^3 = S^3 = (RS)^a(R^{-1}S^{-1})^aRS^{-1} = 1 \rangle$. If $a \equiv 1(\text{mod}3)$ then G is a cyclic group of order 3.

Proof: Taking $H = \langle A = RS, B = R^2S^{-2} \rangle$ as a subgroup of G we define cosets of H in G as $1 = H$, $1R = 2$, $2R = 3$. No collapses occur. We consider coset representatives rather than cosets. Let us put $A = RS$, $B = R^2S^{-2}$. The coset table and coset representative table are respectively :

	R	S	R^{-1}	S^{-1}
1	2	2	3	3
2	3	3	1	1
3	1	1	2	2

	R	S	R^{-1}	S^{-1}
1	2	$A^{-1}2$	3	$B^{-1}3$
2	3	$AB^{-1}3$	1	$A1$
3	1	$B1$	2	$BA^{-1}2$

From the 3 rd relation of G we get only the following nontrivial relations for H

(1) $(AB^{-1}A^{-1}B)^{a/3}(BA^{-1}B^{-1}A)^{a/3}A = 1$,

(2) $(BAB^{-1}A^{-1})^{a/3}(B^{-1}ABA^{-1})^{a/3}BA^{-1} = 1$,

(3) $(A^{-1}BAB^{-1})^{a/3}(ABA^{-1}B^{-1})^{a/3}B^{-1} = 1$.

The relation 3 can be rewritten as $B^{-1}(A^{-1}BAB^{-1})^{a/3}(ABA^{-1}B^{-1})^{a/3} = 1$ ......(*). If we multiply (*) from the right side by $(BAB^{-1}A^{-1})^{a/3}(B^{-1}ABA^{-1})^{a/3}BA^{-1} = 1$ we can get $(A^{-1}BAB^{-1})^{a/3}(B^{-1}ABA^{-1})^{a/3}BA^{-1}B^{-1} = 1$ ......(**). The relation 1 can be rewritten as $A(AB^{-1}A^{-1}B)^{a/3}(BA^{-1}B^{-1}A)^{a/3} = 1$ (***). If we multiply (***) from the right by (**) we can see that H is an abelian subgroup. Since H is an abelian group, from relation 1 we get $A = 1$. Using the fact that $A =1$, from relation 3 we get $B = 1$ and so H is the trivial subgroup. But the index of H in G is 3. Therefore G is a cyclic group of order 3.

Theorem 2.2.3 Let $F = \langle R,S \mid R^3 = S^3 = (RS)^a(R^{-1}S^{-1})^{a+1}RS^{-1} = 1 \rangle$. If $a \equiv 0(\mathrm{mod}3)$ then F is a cyclic group of order 3.

Proof: Let $H = \langle R^2SR^{-2}, RSR^{-1}, RS^{-1}R^{-1}S \rangle$ be a subgroup of F. We can define the cosets $H = 1$, $1R = 2$, $2R = 3$. No collapses occur. We now consider coset representatives rather than cosets. Let us put $A = R^2SR^{-2}$, $B = RSR^{-1}$, $C = RS^{-1}R^{-1}S$. The coset table and coset representative table are respectively :

	R	S	R^{-1}	S^{-1}
1	2	1	3	1
2	3	2	1	2
3	1	3	2	3

	R	S	R^{-1}	S^{-1}
1	2	BC1	3	$C^{-1}B^{-1}1$
2	3	B2	1	$B^{-1}2$
3	1	A1	2	$A^{-1}3$

From the second relation of F we get the first three nontrivial relations for H and from 3 rd relation of F we get the last three nontrivial relations for H as follows

(1) $A^3 = 1$, (4) $(BABC)^{a/3}(A^{-1}B^{-1}C^{-1}B^{-1})^{a/3}A^{-1}C^{-1}B^{-1} = 1$,

(2) $B^3 = 1$, (5) $(ABCB)^{a/3}(C^{-1}B^{-1}A^{-1}B^{-1})^{a/3}C^{-1}B^{-2} = 1$,

(3) $(BC)^3 = 1$, (6) $(BCBA)^{a/3}(B^{-1}C^{-1}B^{-1}A^{-1})^{a/3}B^{-1}A^{-1} = 1$.

The relation 6 can be rewritten as $B^{-1}A^{-1}(BCBA)^{a/3}(B^{-1}C^{-1}B^{-1}A^{-1})^{a/3} = 1$......(i). Multiplying (i) from the left by $(ABCB)^{a/3}(C^{-1}B^{-1}A^{-1}B^{-1})^{a/3}C^{-1}B^{-2} = 1$ and using relation 2 we get $(BCBA)^{a/3}(C^{-1}B^{-1}A^{-1}B^{-1})^{a/3}C^{-1}A^{-1} = 1$ .....(ii). Rewriting relation 4 and taking its inverse we get $(BCBA)^{a/3}(C^{-1}B^{-1}A^{-1}B^{-1})^{a/3}BCA = 1$...(iii). From (ii) and (iii) we can get $C^{-1}A^{-2}C^{-1} = B$. Eliminating B from the presentation of H it becomes

$$
\begin{aligned}
H = \langle A, C \mid\ & A^3 = 1, (C^{-1}A^{-2}C^{-1})^3 = 1, (C^{-1}A^{-2})^3 = 1, \\
& (C^{-1}A^{-2}C^{-1}AC^{-1}A^{-2})^{a/3}(A^{-1}CA^2CA^2C)^{a/3}AC = 1, \\
& (AC^{-1}A^{-2}C^{-1}A^{-2}C^{-1})^{a/3}(A^2CA^{-1}CA^2C)^{a/3}A^2C^2A^2C = 1, \\
& (C^{-1}A^{-2}C^{-1}A^{-2}C^{-1}A)^{a/3}(CA^2CA^2CA^{-1})^{a/3}CA^2CA^{-1} = 1 \ \rangle
\end{aligned}
$$

From the last presentation of H consider the relation 5, rewritten as $A^2C^2A^2C(AC^{-1}A^{-2}C^{-1}A^{-2}C^{-1})^{a/3}(A^2CA^{-1}CA^2C)^{a/3} = 1$ ......(iv). Multiply (iv) from the right side by $(C^{-1}A^{-2}C^{-1}A^{-2}C^{-1}A)^{a/3}(CA^2CA^2CA^{-1})^{a/3}CA^2CA^{-1} = 1$ and use relation 1 from the last presentation of H to get $A^2C^2A^2C^2A^2CA^{-1} = 1$ ....(vi). Relation 2 can be written as $A^2C^2A^2C^2A^2C^2 = 1$ ....(vii). Using (vi) and (vii) together we get $A = C^{-1}$. Therefore H is a cyclic group. Since H is a cyclic group from relation 5 we get $A = 1$ so H is a trivial subgroup. However the index of H in F is 3 and therefore F is a cyclic group of order 3.

Theorem 2.2.4 Let $F = \langle R,S \mid R^3 = S^3 = (RS)^{a+1}(R^{-1}S^{-1})^aRS^{-1} = 1 \rangle$. If $a \equiv 0 (\mathrm{mod} 3)$ then F is a cyclic group of order 3.

Proof: Let $H = \langle R, SRS^{-1}, S^2RS^{-2} \rangle$ be a subgroup of F. We can define the cosets $H = 1$, $1S = 2$, $2S = 3$. No collapses occur. We now consider coset representatives rather than cosets. Let us put $A = R$, $B = SRS^{-1}$, $C = S^2RS^{-2}$.

The coset table and coset representative tables are respectively :

	R	S	R^{-1}	S^{-1}
1	1	2	1	3
2	2	3	2	1
3	3	1	3	2

	R	S	R^{-1}	S^{-1}
1	$A1$	2	$A^{-1}1$	3
2	$B2$	3	$B^{-1}2$	1
3	$C3$	1	$C^{-1}3$	2

From the first relation of F we get the first three nontrivial relations for H and from the 3 rd relation of F we get the last three nontrivial relations for H as follows:

(1) $A^3 = 1$, (4) $(ABC)^{a/3}A(B^{-1}A^{-1}C^{-1})^{a/3}B = 1$,

(2) $B^3 = 1$, (5) $(BCA)^{a/3}B(C^{-1}B^{-1}A^{-1})^{a/3}C = 1$,

(3) $C^3 = 1$, (6) $(CAB)^{a/3}C(A^{-1}C^{-1}B^{-1})^{a/3}A = 1$.

The relations 4 , 5 and 6 can be rewritten as

(4*) $(BCA)^{a/3}(B^{-1}A^{-1}C^{-1})^{a/3}BA = 1$

(5*) $(CAB)^{a/3}(C^{-1}B^{-1}A^{-1})^{a/3}CB = 1$

(6*) $(ABC)^{a/3}(A^{-1}C^{-1}B^{-1})^{a/3}AC = 1$

Rewriting (6*) as $AC(ABC)^{a/3}(A^{-1}C^{-1}B^{-1})^{a/3} = 1$.... (i) and multiplying (i) from the right by (4*) we get $(ABC)^{a/3}(B^{-1}A^{-1}C^{-1})^{a/3}BA^2C = 1$ .... (ii). Taking the inverse of relation (5*) we get $B^{-1}C^{-1}(ABC)^{a/3}(B^{-1}A^{-1}C^{-1})^{a/3} = 1$ ..... (iii). Using (ii), (iii) and relation 3 we can get $B^2A^2 = C$ .... (iv). Eliminating C, the presentation of H will become $H = \langle A, B \mid A^3 = 1, B^3 = 1, (B^2A)^3 = 1, BA = 1, A^2 = 1, B^2 = 1 \rangle$. It is easy to see that H is a trivial group, but the index of H in F is 3. Therefore F is a cyclic group of order 3.

Theorem 2.2.5 Let $F = \langle R,S \mid R^3 = S^3 = (RS)^aRS^{-1} = 1 \rangle$. Then F is a cyclic group of order 3.

Proof: $(RS)^aRS^{-1} = 1 \Rightarrow RS$ commutes with RS^{-1}. Then $RSRS^{-1} = RS^{-1}RS \Rightarrow SRS^{-1} = S^{-1}RS \Rightarrow S^2R = RS^2 \Rightarrow S^{-1}R = RS^{-1} \Rightarrow$ F is abelian. Here we need to consider 3 cases. If $a \equiv 0(\text{mod}3)$ then from relation 3 we get $R = S$. Therefore F is a cyclic group of order 3. If $a \equiv 1(\text{mod}3)$ then from relation 3 we get $R^2 = 1$. Therefore F is a cyclic group of order 3. If $a \equiv 2(\text{mod}3)$ then from relation 3 we get $S = 1$. Therefore F is a cyclic group of order 3.

Theorem 2.2.6 Let $F = \langle R,S \mid R^3 = S^3 = (RS)^a(R^{-1}S^{-1})^b(RS)^c(R^{-1}S^{-1})^d = 1 \rangle$. In the following cases F is an infinite group.

(i) If $(a,b,c,d) = t \geq 3$.

(ii) If $(a,c) = t \geq 3$ and $(b+d,t) = t$.

(iii) If $(b,d) = t \geq 3$ and $(a+c,t) = t$.

Proof: (i) If $(a,b,c,d) = t \geq 3$ then in Theorem 2.1.1 we take $x_i = 0 = y_j$, $i,j = 3,4, \ldots ,m$. The group F will be a special case of F(n) and therefore the proof is obvious by Theorem 2.1.1 **(ii)**.

(ii) If $(a,c) = t \geq 3$ and $(b+d,t) = t$ then adding the relation $(RS)^t = 1$ we can get an infinite homomorphic image of F.

(iii) The proof can be obtained by the same argument as before in **(ii)**.

2.3 RESULTS ON F(4) GROUPS:

The F(4) groups are the groups generated by the two elements R and S of order three subject to one more relation i.e. the groups

$$\langle R,S \mid R^4 = S^4 = w(R,S) = 1 \rangle.$$

We shall show that the derived subgroups of the above groups are perfect groups and moreover some of them are simple groups. Also we give some results on those groups where we have been able to determine their order.

Theorem 2.3.1 If $a \equiv 2 (\mathrm{mod} 4)$ then the derived group of $F = \langle R,S \mid R^4 = S^4 = (RS)^a RS^3R^2S^3 = 1 \rangle$ is a perfect group and has index 4 in F.

Proof: Let $H = \langle R, SRS^{-1}, S^2RS^{-2}, S^3RS^{-3} \rangle$ be a subgroup of F. We can define cosets H = 1, 1S = 2, 2S = 3, 3S = 4. No collapses occur. We now consider coset representatives rather than cosets. Let us put $A = R$, $B = SRS^{-1}$, $C = S^2RS^{-2}$, $D = S^3RS^{-3}$. The coset table and coset representative table are respectively :

	R	S	R^{-1}	S^{-1}
1	1	2	1	4
2	2	3	2	1
3	3	4	3	2
4	4	1	4	3

	R	S	R^{-1}	S^{-1}
1	A1	2	$A^{-1}1$	4
2	B2	3	$B^{-1}2$	1
3	C3	4	$C^{-1}3$	2
4	D4	1	$D^{-1}4$	3

From the first relation of F we get the first four nontrivial relations for H and from the 3 rd relation of F we get the last four nontrivial relations for H as follows :

(1) $A^4 = 1$, (5) $(ABCD)^{(a-2)/4}ABCB^2 = 1$,

(2) $B^4 = 1$, (6) $(BCDA)^{(a-2)/4}BCDC^2 = 1$,

(3) $C^4 = 1$, (7) $(DABC)^{(a-2)/4}DABA^2 = 1$,

(4) $D^4 = 1$, (8) $(CDAB)^{(a-2)/4}CDAD^2 = 1$.

Abelianizing the relations of F we get $| F/F' | = 4$ so the index of F' is 4 but on the other hand the generators of H belong to F' (to see take $F/F' = \langle R,S \mid R =1, S^4 = 1 \rangle$ so $R \in F'$) therefore $H = F'$. Now we are going to show that $F' = H$ is a perfect group. In order to do this we have to consider four cases for $(a-2)/4$.

Case $(a-2)/4 \equiv 0 (mod4)$: Let us assume that $(a-2)/4 \equiv 0(mod4)$. Abelianizing the relations of H and using first four relations of H/H' we get the following relations for H/H' :

(I) $A^4 = 1$, (III) $C^4 = 1$, (V) $ACB^3 = 1$, (VII) $DBA^3 = 1$,

(II) $B^4 = 1$, (IV) $D^4 = 1$, (VI) $BDC^3 = 1$, (VIII) $CAD^3 = 1$

From relation (V) and (VIII) we get $B^3 = D^3$ ..... (i) . From relation (VI) and (VII) we get $A^3 = C^3$ .... (ii). Considering first four relations of H/H' and using (i), (ii) we get $A = C$, $B = D$. Eliminating C and D we get

$H/H' = \langle A, B \mid A^4 = B^4 = A^2B^3 = B^2A^3 = 1 \rangle$, which can easily be seen to be the trivial group. Therefore $H = F'$ is perfect.

Case $(a-2)/4 \equiv 1 (mod4)$: Let us assume that $(a-2)/4 \equiv 1(mod4)$. Abelianizing the relations of H and using the first four relations of H/H' we get the following relations for H/H' :

(I) $A^4 = 1$, (III) $C^4 = 1$, (V) $A^2C^2D = 1$, (VII) $B^2CD^2 = 1$,

(II) $B^4 = 1$, (IV) $D^4 = 1$, (VI) $AB^2D^2 = 1$, (VIII) $A^2BC^2 = 1$

From relation (V) and (VIII) we get $B = D$. From relations (VI) and (VII) we get $A = C$. Eliminating C and D we get the trivial group. Therefore $H = F'$ is a perfect group.

Case $(a-2)/4 \equiv 2 (mod4)$: Let us assume that $(a-2)/4 \equiv 2(mod4)$. Abelianizing the relations of H and using the first four relations of H/H' we get the following relations for H/H' .

(I) $A^4 = 1$, (III) $C^4 = 1$, (V) $A^3BC^3D^2 = 1$, (VII) $AB^3C^2D^3 = 1$,

(II) $B^4 = 1$, (IV) $D^4 = 1$, (VI) $A^2B^3CD^3 = 1$, (VIII) $A^3B^2C^3D = 1$

From relation (V) and (VIII) we get B = D. From relation (VI) and (VII) we get A = C.

Eliminating C and D gives:

$H/H' = \langle A, B \mid A^4 = B^4 = A^2B^3 = B^2A^3 = 1 \rangle$

which can easily be seen to be the trivial group. Therefore H = F' is perfect.

Case (a-2)/4 ≡ 3(mod4): Let us assume that (a-2)/4 ≡ 3(mod4). Abelianizing the relations of H and using the first four relations of H/H' we get the following relations for H/H' :

(I) $A^4 = 1$, (III) $C^4 = 1$, (V) $B^2D^3 = 1$, (VII) $A^2C^3 = 1$,

(II) $B^4 = 1$, (IV) $D^4 = 1$, (VI) $A^3C^2 = 1$, (VIII) $B^3D^2 = 1$

From relation (V) and (VIII) we get B = D. From relation (VI) and (VII) we get A = C. Eliminating C and D we get the trivial group. Therefore H = F' is a perfect group.

Theorem 2.3.2 If a ≡ 1(mod4) then the derived group of

$F = \langle R,S \mid R^4 = S^4 = (RS)^aRS^3R^2S^3 = 1 \rangle$ is a perfect group and has index 4 in F.

Proof: Let $H = \langle S, RSR^{-1}, R^2SR^{-2}, R^3SR^{-3} \rangle$ be a subgroup of F. We can define cosets H = 1, 1R = 2, 2R = 3, 3R = 4. No collapses occur. We now consider coset representatives rather than cosets. Let us put A = S , $B = RSR^{-1}$, $C = R^2SR^{-2}$, $D = R^3SR^{-3}$. The coset table and coset representative table are respectively :

	R	S	R^{-1}	S^{-1}
1	2	1	4	1
2	3	2	1	2
3	4	3	2	3
4	1	4	3	4

	R	S	R^{-1}	S^{-1}
1	2	A1	4	$A^{-1}1$
2	3	B2	1	$B^{-1}2$
3	4	C3	2	$C^{-1}3$
4	1	D4	3	$D^{-1}4$

From the second relation of F we get the first four nontrivial relations for H and from the 3 rd relation of F we get the last four nontrivial relations for H as follows:

(1) $A^4 = 1$, (5) $(BCDA)^{(a-1)/4}BC^3A^3 = 1$,

(2) $B^4 = 1$, (6) $(CDAB)^{(a-1)/4}CD^3B^3 = 1$,

(3) $C^4 = 1$, (7) $(DABC)^{(a-1)/4}DA^3C^3 = 1$,

(4) $D^4 = 1$, (8) $(ABCD)^{(a-1)/4}AB^3D^3 = 1$.

Abelianizing the relations of F we get $|F/F'| = 4$ so the index of F' is 4. On the other hand the generators of H belong to F' so H = F'. Now we are going to show that F' = H is a perfect group. In order to do this we have to consider four cases for (a-1)/4.

Case $(a-1)/4 \equiv 0(\mathrm{mod}4)$: Let us assume that $(a-1)/4 \equiv 0(\mathrm{mod}4)$. Abelianizing the relations of H and using the first four relations of H/H' we get the following relations for H/H' :

(I) $A^4 = 1$, (III) $C^4 = 1$, (V) $BC^3A^3 = 1$, (VII) $DA^3C^3 = 1$,

(II) $B^4 = 1$, (IV) $D^4 = 1$, (VI) $CD^3B^3 = 1$, (VIII) $AB^3D^3 = 1$

From relation (V) and (VII) we get B = D. From relation (VI) and (VIII) we get A = C. Eliminating C and D we get the trivial group. Therefore H = F' is a perfect group.

Case $(a-1)/4 \equiv 1(\mathrm{mod}4)$: Let us assume that $(a-1)/4 \equiv 1(\mathrm{mod}4)$. Abelianizing the relations of H and using the first four relations of H/H' we get the following relations for H/H' :

(I) $A^4 = 1$, (III) $C^4 = 1$, (V) $B^2D = 1$, (VII) $BD^2 = 1$,

(II) $B^4 = 1$, (IV) $D^4 = 1$, (VI) $AC^2 = 1$, (VIII) $A^2C = 1$

From relation (V) and (VII) we get B = D. From relation (VI) and (VIII) we get A = C. Eliminating C and D we get the trivial group. Therefore H = F' is a perfect group.

Case $(a-1)/4 \equiv 2(\mathrm{mod}4)$: Let us assume that $(a-1)/4 \equiv 2(\mathrm{mod}4)$. Abelianizing the relations of H and using the first four relations of H/H' we get the following relations for H/H' :

(I) $A^4 = 1$, (III) $C^4 = 1$, (V) $AB^3CD^2 = 1$, (VII) $AB^2CD^3 = 1$,

(II) $B^4 = 1$, (IV) $D^4 = 1$, (VI) $A^2BC^3D = 1$, (VIII) $A^3BC^2D = 1$.

From relation (V) and (VII) we can get B = D. From relation (VI) and (VIII) we can get A = C. Eliminating C and D we get the trivial group. Therefore H = F' is a perfect group.

Case (a-1)/4 ≡ 3(mod4); Let as assume that (a-1)/4 ≡ 3(mod4). Abelianizing the relations of H and using the first four relations of H/H' we get the following relations for H/H' :

(I) $A^4 = 1$, (III) $C^4 = 1$, (V) $A^2C^2D^3 = 1$, (VII) $A^2B^3C^2 = 1$,

(II) $B^4 = 1$, (IV) $D^4 = 1$, (VI) $A^3B^2D^2 = 1$, (VIII) $B^2C^3D^2 = 1$.

From relation (V) and (VII) we get B = D. From relation (VI) and (VIII) we get A = C. Eliminating C and D we get the trivial group. Therefore H = F' is a perfect group.

Theorem 2.3.3 If a ≡ 0(mod4) then the derived group of $F = \langle R,S \mid R^4 = S^4 = (RS)^a RS^3R^2S^3 = 1 \rangle$ is a perfect group and has index 4 in F.

Proof: Let $H = \langle S^2R, RS^2, S^3RS^{-1}, RSR^{-1}S^{-1} \rangle$ be a subgroup of F. We can define cosets H = 1, 1S = 2, 2S = 3, 3S = 4. No collapses occur. We now consider coset representatives rather than cosets. Let us put $A = S^2R$, $B = RS^2$, $C = S^3RS^{-1}$, $D = RSR^{-1}S^{-1}$. The coset table and coset representative table are respectively :

	R	S	R^{-1}	S^{-1}
1	3	2	3	4
2	4	3	4	1
3	1	4	1	2
4	2	1	2	3

	R	S	R^{-1}	S^{-1}
1	B3	2	$A^{-1}3$	4
2	$D^{-1}B4$	3	$C^{-1}4$	1
3	A1	4	$B^{-1}1$	2
4	C2	1	$B^{-1}D2$	3

From the first relation of F we get the first two nontrivial relations for H, the other two relations are redundant, and from the 3 rd relation of F we get the last four nontrivial relations for H as follows :

(1) $(BA)^2 = 1$, (4) $(D^{-1}B^2CA)^{a/4}D^{-1}BAB = 1$,

(2) $(D^{-1}BC)^2 = 1$, (5) $(AD^{-1}B^2C)^{a/4}ACD^{-1}B = 1$,

(3) $(BCAD^{-1}B)^{a/4}BD^{-1}BC = 1$, (6) $(CAD^{-1}B^2)^{a/4}CBA=1$.

Abelianizing the relations of F we get $|F/F'| = 4$ so the index of F' is 4. On the other hand the generators of H belong to F' so H = F'. Now we are going to show that F' = H is a perfect group. In order to do this we have to consider four cases for a/4.

Case $a/4 \equiv 0(\text{mod}4)$: Let us assume that $a/4 \equiv 0(\text{mod}4)$. Abelianizing the relations of H and consider $a/4 = 4k$, $k \in \mathbb{Z}$. We get the following relations for H/H' :

(I) $A^2B^2 = 1$ (III) $A^{4k}B^{8k+2}C^{4k+1}D^{-4k-1} = 1$ (V) $A^{4k+1}B^{8k+1}C^{4k+1}D^{-4k-1} = 1$

(II) $B^2C^2D^{-2} = 1$ (IV) $A^{4k+1}B^{8k+2}C^{4k}D^{-4k-1} = 1$ (VI) $A^{4k+1}B^{8k+1}C^{4k+1}D^{-4k} = 1$.

Using relations (I) and (II) we can simplify the relations of H/H' as

(i) $A^2B^2 = 1$ (iii) $B^2CD\text{-}1 = 1$ (v) $BCD^{-1}A = 1$

(ii) $B^2C^2D^{-2} = 1$ (iv) $B^2D^{-1}A = 1$ (vi) $ABC = 1$.

From relation (v) and (vi) we get D = 1. From relation (3) and (4) we get A = C. Eliminating C we get $H/H' = \langle A, B \mid A^2B^2 = AB^2 = A^2B = 1 \rangle$, which can easily be seen to be the trivial group. Therefore H = F' is a perfect group.

Case $a/4 \equiv 1(\text{mod}4)$: Let us assume that $a/4 \equiv 1(\text{mod}4)$. Abelianize the relations of H and consider $a/4 = 4k+1$, $k \in \mathbb{Z}$. We get the following relations for H/H' .

(I) $A^2B^2 = 1$ (III) $A^{4k+1}B^{8k+4}C^{4k+2}D^{-4k-2} = 1$ (V) $A^{4k+2}B^{8k+3}C^{4k+2}D^{-4k-2} = 1$

(II) $B^2C^2D^{-2} = 1$ (IV)$A^{4k+2}B^{8k+4}C^{4k+1}D^{-4k-2} = 1$ (VI) $A^{4k+2}B^{8k+3}C^{4k+2}D^{-4k-1} = 1$.

Using relations (I) and (II) we simplify the relations of H/H' as

(i) $A^2B^2 = 1$ (iii) $B^2A = 1$ (v) $B^{-1} = 1$

(ii) $B^2C^2D^{-2} = 1$ (iv) $C^{-1} = 1$ (vi) $BC^2D^{-1} = 1$

From these relations it is easily seen that it is the trivial group. Therefore $H = F'$ is perfect.

Case $a/4 \equiv 2(\mathrm{mod}4)$: Let us assume that $a/4 \equiv 2(\mathrm{mod}4)$. Abelianize the relations of H and consider $a/4 = 4k+2$, $k \in \mathbb{Z}$. We get the following relations for H/H' :

(I) $A^2B^2 = 1$ (III) $A^{4k+2}B^{8k+6}C^{4k+3}D^{-4k-3} = 1$ (V) $A^{4k+3}B^{8k+5}C^{4k+3}D^{-4k-3} = 1$

(II) $B^2C^2D^{-2} = 1$ (IV) $A^{4k+3}B^{8k+6}C^{4k+2}D^{-4k-3} = 1$ (VI) $A^{4k+3}B^{8k+5}C^{4k+3}D^{-4k-2} = 1$.

Using relations (I) and (II) we simplify the relations of H/H' as

(i) $A^2B^2 = 1$ (iii) $C^{-1}D = 1$ (v) $ABCD^{-1} = 1$

(ii) $B^2C^2D^{-2} = 1$ (iv) $AB^2D^{-1} = 1$ (vi) $ABC = 1$.

It can easily be seen that it is the trivial group, therefore $H = F'$ is perfect.

Case $a/4 \equiv 3(\mathrm{mod}4)$: Let us assume that $a/4 \equiv 3(\mathrm{mod}4)$. Abelianize the relations of H and consider $a/4 = 4k+3$, $k \in \mathbb{Z}$. We get the following relations for H/H' :

(I) $A^2B^2 = 1$ (III) $A^{4k+3}B^{8k+8}C^{4k+4}D^{-4k-4} = 1$ (V) $A^{4k+4}B^{8k+7}C^{4k+4}D^{-4k-4} = 1$

(II) $B^2C^2D^{-2} = 1$ (IV) $A^{4k+4}B^{8k+8}C^{4k+3}D^{-4k-4} = 1$ (VI) $A^{4k+4}B^{8k+7}C^{4k+4}D^{-4k-3} = 1$.

Using relations (I) and (II) we simplify the relations of H/H' as

(i) $A^2B^2 = 1$ (iii) $B^2A = 1$ (v) $B^{-1} = 1$

(ii) $B^2C^2D^{-2} = 1$ (iv) $C^{-1} = 1$ (vi) $DB^{-1} = 1$.

From these relations it can easily be seen that it is the trivial group. Therefore $H = F'$ is perfect.

Theorem 2.3.4 If $a \equiv 3(\mathrm{mod}4)$ then the derived group of

$F = \langle R,S \mid R^4 = S^4 = (RS)^aRS^3R^2S^3 = 1 \rangle$ is a perfect group and has index 4 in F.

Proof: Let $H = \langle R^2S, SR^2, R^3SR^{-1}, SRS^{-1}R^{-1} \rangle$ be a subgroup of F. We can define cosets $H = 1$, $1R = 2$, $2R = 3$, $3R = 4$. No collapses occur. We now consider coset representatives rather than cosets. Let us put $A = S^2R$, $B = RS^2$, $C = S^3RS^{-1}$, $D = RSR^{-1}S^{-1}$. The coset table and coset representative table are respectively :

	R	S	R^{-1}	S^{-1}
1	2	3	4	1
2	3	4	1	2
3	4	1	2	3
4	1	2	3	4

	R	S	R^{-1}	S^{-1}
1	2	B3	4	$A^{-1}3$
2	3	$D^{-1}B4$	1	$C^{-1}4$
3	4	A1	2	$B^{-1}1$
4	1	C2	3	$B^{-1}D2$

From the second relation of F we get the first two nontrivial relations for H, the other two relations are redundant, and from the 3 rd relation of F we get the last four nontrivial relations for H as follows :

(1) $(BA)^2 = 1$

(2) $(D^{-1}BC)^2 = 1$

(3) $(D^{-1}B^2CA)^{(a-3)/4+1}BA^2BA = 1$

(4) $(AD^{-1}B^2C)^{(a-3)/4+1}D^{-1}BC^2D^{-1}BC = 1$,

(5) $(CAD^{-1}B^2)^{(a-3)/4+1}AB^2AB = 1$,

(6) $(BCAD^{-1}B)^{(a-3)/4+1}C(D^{-1}B)^2CD^{-1}B = 1$.

Abelianizing the relations of F we get $|F/F'| = 4$ so the index of F' is 4. On the other hand the generators of H belong to F' so H = F'. Now we are going to show that F' = H is a perfect group. In order to do this we have to consider four cases for (a+1)/4.

Case (a+1)/4 ≡ 0(mod4): Let us assume that $(a+1)/4 \equiv 0(\mathrm{mod}4)$. Abelianize the relations of H and consider $(a+1)/4 = 4k$, $k \in \mathbb{Z}$. We get the following relations for H/H' :

(I) $A^2B^2 = 1$ (III) $A^{4k+3}B^{8k+2}C^{4k}D^{-4k} = 1$ (V) $A^{4k+2}B^{8k+3}C^{4k}D^{-4k} = 1$

(II) $B^2C^2D^{-2} = 1$ (IV) $A^{4k}B^{8k+2}C^{4k+3}D^{-4k-2} = 1$ (VI) $A^{4k}B^{8k+3}C^{4k+2}D^{-4k-3} = 1$.

Using relations (I) and (II) we simplify the relations of H/H' as

(i) $A^2B^2 = 1$ (iii) $A = 1$ (v) $B = 1$

(ii) $B^2C^2D^{-2} = 1$ (iv) $C = 1$ (vi) $BD^{-1} = 1$

which can easily be seen to be the trivial group. Therefore H = F' is perfect.

Case (a+1)/4 ≡ 1(mod4): Let us $(a+1)/4 \equiv 1(\mathrm{mod}4)$. Abelianize the relations of H and consider $(a+1)/4 = 4k+1$, $k \in \mathbb{Z}$. We get the following relations for H/H' :

(I) $A^2B^2 = 1$ (III) $A^{4k+4}B^{8k+4}C^{4k+1}D^{-4k-1} = 1$ (V) $A^{4k+3}B^{8k+5}C^{4k+1}D^{-4k-1} = 1$

(II) $B^2C^2D^{-2} = 1$ (IV)$A^{4k+1}B^{8k+4}C^{4k+4}D^{-4k-3} = 1$ (VI) $A^{4k+1}B^{8k+5}C^{4k+3}D^{-4k-4} = 1$.

Using relations (I) and (II) we simplify the relations of H/H' as

(i) $A^2B^2 = 1$ (iii) $CD^{-1} = 1$ (v) $AB^3CD^{-1} = 1$

(ii) $B^2C^2D^{-2} = 1$ (iv) $AD = 1$ (vi) $ABC^{-1} = 1$.

From these relations it can easily be seen that it is the trivial group. Therefore H = F' is perfect.

Case (a+1)/4 ≡ 2(mod4): Let us assume that (a+1)/4 ≡ 2(mod4). Abelianize the relations of H and consider (a+1)/4 = 4k+2, $k \in \mathbb{Z}$. We get the following relations for H/H' :

(I) $A^2B^2 = 1$ (III) $A^{4k+5}B^{8k+6}C^{4k+2}D^{-4k-2} = 1$ (V) $A^{4k+4}B^{8k+7}C^{4k+2}D^{-4k-2} = 1$

(II) $B^2C^2D^{-2} = 1$ (IV) $A^{4k+2}B^{8k+6}C^{4k+5}D^{-4k-4} = 1$ (VI) $A^{4k+2}B^{8k+7}C^{4k+4}D^{-4k-5} = 1$.

Using relations (I) and (II) we simplify the relations of H/H' as

(i) $A^2B^2 = 1$ (iii) $A = 1$ (v) $B = 1$

(ii) $B^2C^2D^{-2} = 1$ (iv) $C = 1$ (vi) $BD^{-1} = 1$.

It can easily be seen that it is the trivial group, therefore H = F' is perfect.

Case (a+1)/4 ≡ 3(mod4): Let us assume that (a+1)/4 ≡ 3(mod4). Abelianize the relations of H and consider (a+1)/4 = 4k+3, $k \in \mathbb{Z}$. We get the following relations for H/H' .

(I) $A^2B^2 = 1$ (III) $A^{4k+6}B^{8k+8}C^{4k+3}D^{-4k-3} = 1$ (V) $A^{4k+5}B^{8k+9}C^{4k+3}D^{-4k-3} = 1$

(II) $B^2C^2D^{-2} = 1$ (IV)$A^{4k+3}B^{8k+8}C^{4k+6}D^{-4k-5} = 1$ (VI) $A^{4k+3}B^{8k+9}C^{4k+5}D^{-4k-6} = 1$

Using relations (I) and (II) we can simplify the relations of H/H' as

(i) $A^2B^2 = 1$ (iii) $CD^{-1} = 1$ (v) $ABC^{-1}D = 1$

(ii) $B^2C^2D^{-2} = 1$ (iv) $AD = 1$ (vi) $ABC^{-1} = 1$.

From these relations it can easily be seen that it is the trivial group. Therefore H = F' is perfect.

Theorem 2.3.5 Let $F = \langle R,S \mid R^4 = S^4 = (RS)^a RS^3R^2S^3 = 1 \rangle$.
If $a \in \{ 0, \mp 1, -2, -3 \}$ then F is a cyclic group of order 4.

Proof: Case a = 0: Let a = 0. Since $a \equiv 0 \pmod 4$ then, by Theorem 2.3.3,

$F' = \langle A, B, C, D \mid (BA)^2 = (D^{-1}BC)^2 = BD^{-1}BC = D^{-1}BAB = ACD^{-1}B = CBA = 1\rangle$

From relations 1 and 4 we get $A = D^{-1}$ ..... (i) . From relations 3 and 4 and using (i) we get $C = D^{-1}$. Eliminating A and C, the presentation of F' will be,

$F' = \langle B, D \mid (BD^{-1})^2 = (D^{-1}BD^{-1})^2 = D^{-3}B = D^{-2}B = 1 \rangle$. From relations 3 and 4 in the last presentation of F' we get D = 1. Therefore F' is trivial so F is an abelian group. Abelianizing the relations of F we see that F is a cyclic group of order 4 .

Case a = -1: Let a = -1. Since $a \equiv 3 \pmod 4$ then, by Theorem 2.3.4,

$$F' = \langle A, B, C, D \mid (BA)^2 = (D^{-1}BC)^2 = BA^2BA = D^{-1}BC^2D^{-1}BC = AB^2AB$$
$$= C(D^{-1}B)^2CD^{-1}B = 1\rangle.$$

From relations 1 and 3 we get A = 1..... (i) From relations 1 and 5 and using (i) we get B = 1.... (ii) . From relations 2 and 4 and using (ii) we get D =1...(iii).

Using (i), (ii), (iii) we can see that F' is trivial so F is an abelian group, abelianizing the relations of F we see that F is a cyclic group of order 4 .

Case a = 1: Let a = 1. Since $a \equiv 1 \pmod 4$ then, by Theorem 2.3.2,

$$F' = \langle A, B, C, D \mid A^4 = B^4 = C^4 = D^4 =1,$$
$$BC^3A^3 = CD^3B^3 = DA^3C^3 = AB^3D^3 = 1\rangle.$$

From relation 5, using relations 1 and 3, we get B = AC..... (i). From relation 7, using relations 1, 3 and using (i), we get D = CA.... (ii) . Eliminating B and D the presentation of F' will be:

$F' = \langle A, C \mid A^4 = 1, (AC)^4 = 1, C^4 = 1, C(CA)^3(AC)^3 = A(AC)^3(CA)^3 = 1 \rangle$. From relation 4 and using relation 2 we get $CA^{-1}C^{-2}A^{-1} =1$...(iii). From relation 5, using relation 2, we get $AC^{-1}A^{-2}C^{-1} =1$...(iv). Now multiplying (iii) by (iv) and using relations 1 and 3 we get $A = C^{-1}$.....(v). Using (v) we can see that F' is trivial so F is an abelian group. Abelianizing the relations of F we see that F is a cyclic group of order 4 .

Case a = -2: Let a =-2. Since a $\equiv 2 (\bmod 4)$ then, by Theorem 2.3.1,

$$F' = \langle A, B, C, D \mid A^4 = B^4 = C^4 = D^4 = 1,$$
$$D^{-1}B^2 = A^{-1}C^2 = C^{-1}A^2 = B^{-1}D^2 = 1 \rangle.$$

From relation 5 we get $D = B^2$..... (i) . Using (i) in relation 8 we get B =1 ...(ii). Eliminating B and D we can easily see that F' is trivial so F is an abelian group. Abelianizing the relations of F we see that F is a cyclic group of order 4 .

Case a = -3; Let a = -3. Since a $\equiv 1 (\bmod 4)$ then, by Theorem 2.3.2,

$$F' = \langle A, B, C, D \mid A^4 = B^4 = C^4 = D^4 = 1,$$
$$A^2D^{-1}C^2 = B^2A^{-1}D^2 = C^2B^{-1}A^2 = D^2C^{-1}B^2 = 1 \rangle.$$

From relation 5 we get $D = C^2A^2$..... (i). From relation 7 we get $B = A^2C^2$ ... (ii). Eliminating B and D it can easily be seen that F' is trivial, so F is an abelian group. Abelianizing the relations of F we see that F is a cyclic group of order 4.

Theorem 2.3.6 Let $F = \langle R,S \mid R^4 = S^4 = (RS)^aRS^3R^2S^3 = 1 \rangle$.

(i) If a = 2 or -5 then F' = PSL(2,7).

(ii) If a = 3 or -6 then F' = PSL(2,17).

Proof (i) : Case a = 2: Let a = 2. Since a $\equiv 2 (\bmod 4)$ then, by Theorem 2.3.1,

$$F' = \langle A, B, C, D \mid A^4 = B^4 = C^4 = D^4 = 1,$$
$$ABCB^2 = BCDC^2 = DABA^2 = CDAD^2 = 1 \rangle.$$

Let $K = \langle B^{-1}D^{-1}, BC \rangle$ be a subgroup of F'. The index of K in F' is 8. We define cosets K = 1, 1B = 2, 1D = 3, 2B = 4, 1C = 5, 4C =6, 6B = 7, 2D =8. No collapses occur. We now consider coset representatives rather than cosets. Let us put $X = B^{-1}D^{-1}$, $Y = BC$. The coset table and coset representatives are respectively :

	A	B	C	D	A^{-1}	B^{-1}	C^{-1}	D^{-1}
1	6	2	5	3	4	3	2	5
2	7	4	1	8	5	1	3	7
3	4	1	2	6	6	4	5	1
4	1	3	6	7	3	2	7	8
5	2	8	3	1	8	7	1	6
6	3	7	8	5	1	8	4	3
7	8	5	4	2	2	4	8	4
8	5	6	7	4	7	6	6	2

Coset representatives:

$1A = X^{-1}Y^3X^{-1}Y^2X^{-1}Y^{-1}6$ $\quad$ $5A = Y^{-1}XY^{-2}XY^{-1}2$

$2A = YX^{-1}Y^2X^{-1}YXY^{-1}X^{-1}Y^3X^{-1}Y^2X^{-1}Y^{-1}7$ $\quad$ $6A = YXY^{-2}XY^{-1}3$

$3A = X^{-1}Y^2X^{-1}4$ $\quad$ $7A = YXY^{-2}XY^{-3}XYXY^{-2}XY^{-3}XYXY^{-2}XY^{-3}X2Y^{-2}XY^{-1}8$

$4A = XY^{-2}XY^{-2}X1$ $\quad$ $8A = YX^{-1}Y^2X^{-2}Y^3X^{-1}Y^2X^{-1}Y^{-1}X^{-1}Y^3X^{-1}Y^2X^{-2}5$

$1A^{-1} = X^{-1}Y^2X^{-1}Y^2X^{-1}4$ $\quad$ $5A^{-1} = X^2Y^{-2}XY^{-3}XYXY^{-2}XY^{-3}X^2Y^{-2}XY^{-1}8$

$2A^{-1} = YX^{-1}Y^2X^{-1}Y5$ $\quad$ $6A^{-1} = YXY^{-2}XY^{-3}X1$

$3A^{-1} = YX^{-1}Y^2X^{-1}Y^{-1}6$ $\quad$ $4A^{-1} = XY^{-2}X3$ $\quad$ $7A^{-1} = YXY^{-2}XY^{-3}X1$

$8A^{-1} = YX^{-1}Y^2X^{-2}Y^3X^{-1}Y^2X^{-1}Y^{-1}X^{-1}Y^3X^{-1}Y^2X^{-1}Y^{-1}X^{-1}Y^3X^{-1}Y^2X^{-1}Y^{-1}7$

$1B = 2$ $\quad$ $2B = 4$ $\quad$ $3B = X^{-1}1$ $\quad$ $4B = X3$

$5B = X^2Y^{-2}XY^{-3}XYXY^{-2}XY^{-3}XYX^2Y^{-2}XY^{-3}XYXY^{-2}XY^{-3}X^2Y^{-2}XY^{-1}$ $\quad$ $6B = 7$

$7B = YXY^{-2}XY^{-3}XYX^{-1}Y^{-1}XY^{-2}XY^{-1}XY^{-1}5$

$8B = YX^{-1}Y^2X^{-2}Y^3X^{-1}Y^2X^{-1}Y^{-1}X^{-1}Y^3X^{-1}Y^2X^{-2}Y^2X^{-1}Y^2X^{-1}Y^{-1}X^{-1}Y^3X^{-1}Y^2$
$\quad X^{-2}YX^{-1}YX^{-1}Y^2X^{-1}YXY^{-1}X^{-1}Y^3X^{-1}Y^2X^{-1}Y^{-1}6$

$1B^{-1} = X3$ $\quad$ $2B^{-1} = 1$ $\quad$ $3B^{-1} = X^{-1}4$ $\quad$ $4B^{-1} = 2$

$5B^{-1} = YX^{-1}YX^{-1}Y^2X^{-1}YXY^{-1}X^{-1}Y^3X^{-1}Y^2X^{-1}Y^{-1}7$

$6B^{-1} = YXY^{-2}XY^{-3}XYX^{-1}Y^{-1}XY^{-2}XY^{-1}XY^{-1}X^2Y^{-2}XY^{-3}XYXY^{-2}XY^{-3}XYX^2$
$\quad Y^{-2}XY^{-3}XYXY^{-2}XY^{-3}X^2Y^{-2}XY^{-1}8$ $\quad$ $7B^{-1} = 6$

$8B^{-1} = YX^{-1}Y^2X^{-1}Y^3X^{-1}Y^2X^{-1}Y^{-1}X^{-1}Y^3X^{-1}Y^2X^{-2}Y^{-1}X^{-1}Y^3X^{-1}Y^2X^{-1}Y^{-1}X^{-1}Y^3$

$X^{-1}Y^2X^{-2}5$ $\quad 1C = 5 \quad 2C = Y1 \quad 3C = Y^{-2}2 \quad 4C = 6$

$5C = Y3 \quad 6C = XY^{-2}8 \quad 7C = YXY^{-2}XY^{-3}X4 \quad 8C = Y^2X^{-2}Y^3X^{-1}Y7$

$1C^{-1} = Y^{-2}2 \quad 2C^{-1} = Y^23 \quad 3C^{-1} = Y^{-1}5 \quad 4C^{-1} = X^{-1}Y^3X^{-1}Y^2X^{-1}Y^{-1}7$

$5C^{-1} = 1 \quad 6C^{-1} = 4 \quad 7C^{-1} = YXY^{-2}XY^{-3}X^2Y^{-2}8 \quad 8C^{-1} = Y^2X^{-1}6$

$1D = 3 \quad 2D = 8 \quad 3D = YX^{-1}Y^{-1}6$

$4D = X^{-1}Y^3X^{-1}Y^2X^{-1}Y^{-1}X^{-1}Y^3X^{-1}Y^2X^{-1}Y^{-1}7 \quad 5D = XY^{-1}1$

$6D = Y5 \quad 7D = YXY^{-2}XY^{-3}XYXY^{-2}XY^{-3}X^2Y^{-2}XY^{-1}2 \quad 8D = YX^{-1}Y^2X^{-1}4$

$1D^{-1} = YX^{-1}5 \quad 2D^{-1} = YX^{-1}Y^2X^{-2}Y^3X^{-1}Y^2X^{-1}Y^{-1}X^{-1}Y^3X^{-1}Y^2X^{-1}Y^{-1}7$

$3D^{-1} = 1 \quad 4D^{-1} = XY^{-2}XY^{-1}8 \quad 5D^{-1} = Y^{-1}6 \quad 6D^{-1} = YXY^{-1}3$

$7D^{-1} = YXY^{-2}XY^{-3}XYXY^{-2}XY^{-3}X4 \quad 8D^{-1} = 2$

Let us denote relations of K as V_i, $i = 1, 2, 3, ..., 17$. From the following we can get the nontrivial relations for K.

$5BCDC^2 = 5 \Rightarrow V_1$	$7ABCB^2 = 7 \Rightarrow V_{10}$
$7BCDC^2 = 7 \Rightarrow V_2$	$8ABCB^2 = 8 \Rightarrow V_{11}$
$8BCDC^2 = 8 \Rightarrow V_3$	$8CDAD^2 = 8 \Rightarrow V_{12}$
$1CDAD^2 = 1 \Rightarrow V_4$	$3DABA^2 = 3 \Rightarrow V_{13}$
$3CDAD^2 = 3 \Rightarrow V_5$	$4DABA^2 = 4 \Rightarrow V_{14}$
$6CDAD^2 = 6 \Rightarrow V_6$	$6DABA^2 = 6 \Rightarrow V_{15}$
$4ABCB^2 = 4 \Rightarrow V_7$	$7DABA^2 = 7 \Rightarrow V_{16}$
$5ABCB^2 = 5 \Rightarrow V_8$	$8DABA^2 = 8 \Rightarrow V_{17}$
$6ABCB^2 = 6 \Rightarrow V_9$	

Now we can write down the relations of K

$V_1 = X^3Y^{-2}XY^{-3}XYXY^{-2}XY^{-3}XYX^2Y^{-2}XY^{-3}XYXY^{-2}XY^{-3}X^2Y^{-2}XYX^{-1}Y$
$XY^{-2}XY^{-3}X^2Y^{-2}$

$V_2 = X^{-2}Y^{-1}XY^{-2}XY^{-1}XY$

$V_3 = YX^{-1}Y^2X^{-2}Y^3X^{-1}Y^2X^{-1}Y^{-1}X^{-1}Y^3X^{-1}Y^2X^{-2}Y^{-1}X^{-1}Y^3X^{-1}Y^2X^{-1}Y^{-1}X^{-1}Y^3$

$X^{-1}Y^2X^{-2}YX^{-1}YX^{-1}Y^2X^{-1}YXY^{-1}X^{-1}Y^3X^{-1}Y^2X^{-1}Y^{-1}XY^{-1}X^{-1}$

$V_4 = XY^{-1}X^{-1}Y^3X^{-1}Y$

$V_5 = Y^{-2}X^{-1}Y^2X^{-2}Y^3X^{-1}Y^2X^{-1}Y^{-1}X^{-1}Y^3X^{-1}Y^2X^{-1}$

$V_6 = XY^{-3}XYX^{-1}Y^{-1}$

$V_7 = XY^{-2}XY^{-2}XY$

$V_8 = Y^{-2}XY^{-2}X^2Y^{-2}XY^{-3}XYX^{-1}Y^{-1}XY^{-2}XY^{-1}X$

$V_9 = Y^2X^{-1}YX^{-1}YX^{-1}Y^2X^{-1}YXY^{-1}X^{-1}$

$V_{10} = X^{-2}Y^3X^{-1}Y^2X^{-1}Y^{-1}X^{-1}Y^3X^{-1}Y^2X^{-2}YX^{-1}YX^{-1}Y^2X^{-1}YXY^{-1}X^{-1}Y^3X^{-1}Y^2X^{-1}$

$Y^{-1}XY^{-1}X^{-1}Y^2X^{-2}Y^3X^{-1}Y^2X^{-1}Y^{-1}X^{-1}Y^3X^{-1}Y^2X^{-2}Y^{-1}X^{-1}Y^3X^{-1}Y^2X^{-1}$

$Y^{-1}X^{-1}Y^3X^{-1}Y^2X^{-2}YX^{-1}YX^{-1}Y^2$

$V_{11} = XYX^2Y^{-2}XY^{-3}XYXY^{-2}XY^{-3}X^2Y^{-2}XYX^{-1}YX^{-1}Y^{-1}XY^{-2}XY^{-1}XY^{-1}X^2Y^{-2}$

$XY^{-3}XYXY^{-2}XY^{-3}XYX^2Y^{-2}XY^{-3}XYXY^{-2}XY^{-3}$

$V_{12} = YX^{-1}YXY^{-2}XY^{-3}XYX^2Y^{-2}XY^{-3}X^2Y^{-2}X$

$V_{13} = YXY^{-1}X^{-2}$

$V_{14} = X^{-2}Y^3X^{-1}Y^2X^{-2}Y^{-1}X^{-1}Y^3X^{-1}Y^2X^{-1}Y^{-1}X^{-1}Y^3X^{-1}Y^2X^{-2}YX^{-1}YX^{-1}$

$Y^2X^{-1}YXY^{-1}X^{-1}Y^2X^{-1}Y^2$

$V_{15} = XY^{-2}XY^{-1}XY^{-2}XYX^{-1}Y^2X^{-1}Y^{-1}$

$V_{16} = X^2Y^{-2}XY^{-3}X^2Y^{-2}XY^{-1}XY^{-1}$

$V_{17} = Y^{-2}XYX^{-1}Y^2X^{-1}YX^2Y^{-2}XY^{-3}XYXY^{-2}XY^{-3}X^2Y^{-2}X$

There are no further nontrivial relation. Now we can write the presentation for K as:

$K = \langle X, Y \mid V_i = 1, \quad 1 \leq i \leq 17 \rangle$.

From relation V_{13} we get $YX = X^2Y$ ..... (i). Using (i) in the relation V_4 we get $Y^3 = 1$ .... (ii). Using (ii) in V_2 we get $X^7 = 1$ ... (iii). Using (i), (ii), (iii) the relations of K can be simplified. After simplification the presentation of K will have the form

$K = \langle X, Y \mid X^7 = 1, \; Y^3 = 1, \; YX = X^2Y \rangle$. K is a semi direct product $(C_7{:}C_3)$ and order of K is 21, but the index of K in F' is 8, therefore $|F'| = 168$. By

Theorem 2.3.1, F' is a perfect group. The only simple group of order dividing 168 is PSL(2,7) so F' is isomorphic to PSL(2,7). Obviously the order of F is 672, since the index of F' in F is 4 by Theorem 2.3.1.

Case a = -5: Let a = -5. Since a $\equiv 3 (\mathrm{mod} 4)$ then, by Theorem 2.3.4,

$$F' = \langle A, B, C, D \mid (BA)^2 = 1, (D^{-1}BC)^2 = 1, (D^{-1}B^2CA)^{-1}BA^2BA = 1,$$
$$(AD^{-1}B^2C)^{-1}D^{-1}BC^2D^{-1}BC = 1, (CAD^{-1}B^2)^{-1}AB^2AB = 1,$$
$$(BCAD^{-1}B)^{-1}C(D^{-1}B)^2CD^{-1}B = 1 \rangle$$

Using the TC program (by machine) we see that the order of F' is 168. By Theorem 2.3.4 F' is a perfect group. The only simple group dividing that order is PSL(2,7) therefore F' is isomorphic to PSL(2,7). Obviously the order of F is 672, since the index of F' is 4 by Theorem 2.3.4.

Case a = 3: Let a = 3. Since a $\equiv 3 (\mathrm{mod} 4)$ then, by Theorem 2.3.4,

$$F' = \langle A, B, C, D \mid (BA)^2 = 1, (D^{-1}BC)^2 = 1, D^{-1}B^2CABA^2BA = 1,$$
$$AD^{-1}B^2CD^{-1}BC^2D^{-1}BC = 1, CAD^{-1}B^2AB^2AB = 1,$$
$$CAD^{-1}BC(D^{-1}B)^2CD^{-1}B^2 = 1 \rangle.$$

Using the TC program (by machine) we see that the order of F' is 2448. By Theorem 2.3.4 F' is a perfect group. The only simple group dividing that order is PSL(2,17). Therefore F' is isomorphic to PSL(2,17). Obviously the order of F is 9792, since the index of F' is 4 by Theorem 2.3.4.

Case a = -6: Let a = -6. Since a $\equiv 2 (\mathrm{mod} 4)$ then by Theorem 2.3.1

$$F' = \langle A, B, C, D \mid A^4 = B^4 = C^4 = D^4 = 1, (ABCD)^{-2}ABCB^2 = 1$$
$$(BCDA)^{-2}BCDC^2 = 1, (DABC)^{-2}DABA^2 = 1, CDAB)^{-2}CDAD^2 = 1\rangle.$$

Using the TC program (by machine) we see that the order of F' is 2448. By Theorem 2.3.1 F' is a perfect group. The only simple group dividing that order is PSL(2,17). Therefore F' is isomorphic to PSL(2,17). Obviously the order of F is 9792, since the index of F' is 4 by Theorem 2.3.1.

2.4 A CLASS OF GROUPS OF ORDER 2

and

A CLASS OF GROUPS OF ORDER $4.(2^{n/2}-1)^2$

In this section we investigate the structure of the groups of two different classes whose presentation was mentioned in the introduction of this chapter. The groups in the first class turn out to be the cyclic group of order 2 and the groups in the second class turn out to be metabelian groups of order $4.(2^{n/2}-1)^2$. Moreover the derived group of the groups in the second class is the direct product of two copies of a cyclic group of order $2^{n/2}-1$.

There is connection between (4,n) - Groups and these two classes. Take

$G = \langle a,b \mid a^4 = b^2 = a^2ba^2bab^{-1}ab^{-1} = 1 \rangle$. Now take

$F = \langle a_1 = a, a_2 = bab^{-1}, a_3 = b^2ab^{-2}, \ldots , a_n = bn^{n-1}ab^{1-n} \rangle$ as a subgroup of G. The index of F in G is n. Using R.S. method, F has the following presentation

$F = \langle a_1, a_2, a_3, \ldots , a_n \mid a_i^4 = 1,\ a_i^2 a_{i+1}^2 a_{i+2} a_{i+1} = 1,\ 1 \le i \le n \rangle$

(i) If $n \ge 5$ and if n is odd then F can be generated by $a_1 = A$ and $a_2 = B$ and has the presentation given in Theorem 2.4.1 .

(ii) If $n \ge 6$ and if n is even then F can be generated by $a_1 = A$ and $a_2 = B$ and has the presentation given in Theorem 2.4.2 .

Theorem 2.4.1 Let $n \ge 5$ and n be an odd integer. The group F generated by A and B and subject to the following relations is a cyclic group of order 2.

$A^4 = 1$,

$B^4 = 1$,

$(B(AB)^2)^4 = 1$,

$(B(BA)^6)^4 = 1$,

$(B(BA)^{14})^4 = 1$,

"

$(B(BA)^{(2^{(n-1)/2}-2)})^4 = 1,$

$(A^{-1}B^{-1})^{2^{(n-3)/2}}B(BA)^{(2^{(n-1)/2}-2)}B(BA)^{2^{(n-3)/2}}B^{-1}(A^{-1}B^{-1})^{(2^{(n-1)/2}-2)}B^{-1}$

$(A^{-1}B^{-1})^{(2^{(n+1)/2}-3)}A(BA)^{(2^{(n-1)/2}-1)}B^{-1} = 1,$

$(BA)^{2^{(n-1)/2}}B^{-1}(A^{-1}B^{-1})^{(2^{(n-1)/2}-2)}B^{-1}(A^{-1}B^{-1})^{(2^{(n+1)/2}-3)}A^2 = 1.$

Proof: Let $H = \langle BA^{-1}, AB, A^2 \rangle$ to be a subgroup of F. Then we define cosets $H = 1$, $1A = 2$. No collapses occur. We now consider coset representatives rather than cosets. Let us put $C = BA^{-1}$, $D = AB$, $E = A^2$. The coset table and coset representative table are respectively :

	A	B	A^{-1}	B^{-1}
1	2	2	2	2
2	1	1	1	1

	A	B	A^{-1}	B^{-1}
1	2	C2	$E^{-1}2$	$D^{-1}2$
2	E1	D1	1	$C^{-1}1$

Using the modified TC algorithm, from the first (n+3)/2 relations in F using coset 1 as a coset representative the following relations were obtained

$E^2 = 1,$

$(CD)^2 = 1,$

$(D^3(CE)^2C)^2 = 1,$

$(D^7(CE)^6C)^2 = 1,$

"

"

$(D^{(2^{(n-1)/2}-1)}(CE)^{(2^{(n-1)/2}-2)}C)^2 = 1,$

$D^{(2^{(n-1)/2}-1)}(C^{-1}E^{-1})^{2^{(n-3)/2}}D^{(2^{(n-1)/2}-1)}(CE)^{2^{(n-3)/2}}D^{-(2^{(n-1)/2}-1)}$

$(C^{-1}E^{-1})^{(2^{(n+1)/2}-3)}C^{-1} = 1.$

From relation (n+5)/2 in F using coset 1 as coset representative $D^{-(2^{(n-1)/2}-1)}(C^{-1}E^{-1})^{(2^{(n+1)/2}-3)}C^{-1}(EC)^{2^{(n-1)/2}}E = 1$ is the relation obtained .

From the first (n+1)/2 relations in F using coset 2 as coset representative we get redundant relations respectively of the first (n+1)/2 relations.

From relations (n+3)/2 and (n+5)/2 in F using coset 2 as coset representative we have the following relations respectively

$D^{-2^{(n-3)/2}}(CE)^{(2^{(n-1)/2}-2)}CD^{2^{(n-3)/2}}(C^{-1}E^{-1})^{(2^{(n-1)/2}-2)}C^{-1}D^{-(2^{(n+1)/2}-2)}$
$(EC)^{(2^{(n-1)/2}-1)}E = 1,\ D^{2^{(n-1)/2}}(C^{-1}E^{-1})^{(2^{(n-1)/2}-2)}C^{-1}D^{-(2^{(n+1)/2}-2)}E = 1.$

Now we can write down the relations of H.

$E^2 = 1,$

$(CD)^2 = 1,$

$(D^3(CE)^2C)^2 = 1,$

$(D^7(CE)^6C)^2 = 1,$

"

"

$(D^{(2^{(n-1)/2}-1)}(CE)^{(2^{(n-1)/2}-2)}C)^2 = 1,$
$D^{(2^{(n-1)/2}-1)}(C^{-1}E^{-1})^{2^{(n-3)/2}}D^{(2^{(n-1)/2}-1)}(CE)^{2^{(n-3)/2}}D^{-(2^{(n-1)/2}-1)}(C^{-1}E^{-1})^{(2^{(n+1)/2}-3)}$
$C^{-1} = 1,$
$D^{-2^{(n-3)/2}}(CE)^{(2^{(n-1)/2}-2)}CD^{2^{(n-3)/2}}(C^{-1}E^{-1})^{(2^{(n-1)/2}-2)}C^{-1}D^{-(2^{(n+1)/2}-2)}$
$(EC)^{(2^{(n-1)/2}-1)}E = 1,$
$D^{-(2^{(n-1)/2}-1)}(C^{-1}E^{-1})^{(2^{(n+1)/2}-3)}C^{-1}(EC)^{2^{(n-1)/2}}E = 1,$
$D^{2^{(n-1)/2}}(C^{-1}E^{-1})^{(2^{(n-1)/2}-2)}C^{-1}D^{-(2^{(n+1)/2}-2)}E = 1.$

Now we shall show that H is the trivial group.

Consider relation (n+9)/2 i.e.
$D^{2^{(n-1)/2}}(C^{-1}E^{-1})^{(2^{(n-1)/2}-2)}C^{-1}D^{-(2^{(n+1)/2}-2)}E = 1$. We can write it as
$D^{2^{(n-1)/2}}(C^{-1}E^{-1})^{(2^{(n-1)/2}-2)}C^{-1}D^{-(2^{(n+1)/2}-2)} = E$, since $E^2 = 1$.

Squaring both sides of this relation and taking its inverse gives
$(D^{(2^{(n-1)/2}-2)}(CE)^{(2^{(n-1)/2}-2)}C)^2 = 1$ ...... (i).

Now consider the relation $(n+1)/2$ i.e.

$(D^{(2^{(n-1)/2}-1)}(CE)^{(2^{(n-1)/2}-2)}C)^2 = 1$. Rewrite it as

$D(D^{(2^{(n-1)/2}-2)}(CE)^{(2^{(n-1)/2}-2)}C)D^{(2^{(n-1)/2}-2)}D(CE)^{(2^{(n-1)/2}-2)}C = 1$. Using (i) we get

$D(CE)^{(2^{(n-1)/2}-2)}C = C(EC)^{(2^{(n-1)/2}-2)}D^{-1}$ .... (ii).

Now consider again the relation $(n+1)/2$ i.e.

$(D^{(2^{(n-1)/2}-1)}(CE)^{(2^{(n-1)/2}-2)}C)^2 = 1$. Using (ii) we get $(CE)^{(2^{(n-1)/2}-2)}C)^2 = 1$ ....(iii).

Using relation $(n+5)/2$ i.e.

$D^{-2^{(n-3)/2}}(CE)^{(2^{(n-1)/2}-2)}CD^{2^{(n-3)/2}}(C^{-1}E^{-1})^{(2^{(n-1)/2}-2)}C^{-1}D^{-(2^{(n+1)/2}-2)}$

$(EC)^{(2^{(n-1)/2}-1)}E = 1$, and using (iii) in this relation, we get

$D^{(-3.2^{(n-1)/2}+2)}(EC)^{(2^{(n-1)/2}-1)}E = 1$ .... (iv).

Consider relation $(n+9)/2$ i.e.

$D^{2^{(n-1)/2}}(C^{-1}E^{-1})^{(2^{(n-1)/2}-2)}C^{-1}D^{-(2^{(n+1)/2}-2)}E = 1$. Using (ii) in this relation we get

$D^{(-3.2^{(n-1)/2}+2)}(EC)^{(2^{(n-1)/2}-1)} = 1$ ...... (v). From (iv) and (v) it is obvious that

$E = 1$ .... (vi).

Consider relation $(n+7)/2$ i.e.

$D^{-(2^{(n-1)/2}-1)}(C^{-1}E^{-1})^{(2^{(n+1)/2}-3)}C^{-1}(EC)^{2^{(n-1)/2}}E = 1$. Using (vi) in this relation

we get $D^{(-2^{(n-1)/2}+1)}C^{(-2^{(n-1)/2}+2)} = 1$ .... (vii).

Consider relation $(n+9)/2$ i.e.

$D^{2^{(n-1)/2}}(C^{-1}E^{-1})^{(2^{(n-1)/2}-2)}C^{-1}D^{-(2^{(n+1)/2}-2)}E = 1$. Using (vi) we get

$D^{(-2^{(n-1)/2}+2)}C^{(-2^{(n-1)/2}+1)} = 1$ ...... (viii). From (vii) and (viii) we see that

$C = D$ .... (ix).

Using (ix) in (viii) and considering relation 1 we can see that $D= 1$, therefore H is the trivial group. But the index of H in F is 2. Therefore F is a cyclic group of order 2.

Theorem 2.4.2 Let $n \geq 6$ be an even integer and let F be the group generated by A and B and subject to the following relations then

(i) $F' = C_{(2^{n/2}-1)} \times C_{(2^{n/2}-1)}$

(ii) order of F is $4.(2^{n/2}-1)^2$.

The relations are:

$A^4 = 1,$

$B^4 = 1,$

$(B(AB)^2)^4 = 1,$

$(B(BA)^6)^4 = 1,$

$(B(BA)^{14})^4 = 1$

"

"

$(B(BA)^{(2^{n/2}-2)})^4 = 1,$

$B^{-1}(BA)^{2^{(n-2)/2}}B(BA)^{(2^{n/2}-2)}B(A^{-1}B^{-1})^{(2^{(n-2)/2}-1)}A(BA)^{(2^{n/2}-1)} = 1,$

$(BA)^{(2^{n/2}+2^{(n-2)/2}+2)}B(BA)^{(2^{n/2}-2)}B(A^{-1}B^{-1})^{(2^{(n-2)/2}-1)}A^2 = 1.$

Proof: (i) Let $H = \langle AB, A^2BA^{-1}, A^3BA^{-2}, BA^{-3} \rangle$ be a subgroup of F. We define cosets H = 1, 1A = 2. 2A = 3, 3A = 4. No collapses occur. We now consider coset representatives rather than cosets. Let us put $X = AB$, $Y = A^2BA^{-1}$, $Z = A^3BA^{-2}$, $T = BA^{-3}$. The coset table and coset representative table are respectively :

	A	B	A^{-1}	B^{-1}
1	2	4	4	2
2	3	1	1	3
3	4	2	2	4
4	1	3	3	1

	A	B	A^{-1}	B^{-1}
1	2	T4	4	$X^{-1}2$
2	3	X1	1	$Y^{-1}1$
3	2	Y2	2	$Z^{-1}2$
4	1	Z3	1	$T^{-1}1$

Using the modified TC algorithm, from the first (n+2)/2 relations in F using coset 1 as coset representative the following relations were obtained respectively :

$TZYX = 1,$

$X^3T^3Z^3Y^3 = 1,$

$X^7T^7Z^7Y^7 = 1,$

$X^{15}T^{15}Z^{15}Y^{15} = 1,$

$X^{31}T^{31}Z^{31}Y^{31} = 1,$

$X^{63}T^{63}Z^{63}Y^{63} = 1,$

"

$X^{(2^{n/2}-1)}T^{(2^{n/2}-1)}Z^{(2^{n/2}-1)}Y^{(2^{n/2}-1)} = 1.$

From the first (n+2) relations in F using cosets 2, 3 and 4 as coset representatives we get redundant relations. From relation (n+4)/2 in F using cosets 1, 2, 3 and 4 as coset representatives we have the following relations for H :

$X^{2^{(n-2)/2}}T^{(2^{n/2}-1)}Z^{-(2^{(n-2)/2}-1)}T^{(2^{n/2}-1)} = 1,$

$Y^{2^{(n-2)/2}}X^{(2^{n/2}-1)}T^{-(2^{(n-2)/2}-1)}X^{(2^{n/2}-1)} = 1,$

$Z^{2^{(n-2)/2}}Y^{(2^{n/2}-1)}X^{-(2^{(n-2)/2}-1)}Y^{(2^{n/2}-1)} = 1,$

$T^{2^{(n-2)/2}}Z^{(2^{n/2}-1)}Y^{-(2^{(n-2)/2}-1)}Z^{(2^{n/2}-1)} = 1.$

From relation (n+6)/2 in F, using cosets 1, 2, 3 and 4 as coset representatives, we get the following relations :

$T^{(2^{n/2}+2^{(n-2)/2}-1)}Z^{(2^{n/2}-1)}Y^{-(2^{(n-2)/2}-1)} = 1,$

$X^{(2^{n/2}+2^{(n-2)/2}-1)}T^{(2^{n/2}-1)}Z^{-(2^{(n-2)/2}-1)} = 1,$

$Y^{(2^{n/2}+2^{(n-2)/2}-1)}X^{(2^{n/2}-1)}T^{-(2^{(n-2)/2}-1)} = 1,$

$Z^{(2^{n/2}+2^{(n-2)/2}-1)}Y^{(2^{n/2}-1)}X^{-(2^{(n-2)/2}-1)} = 1.$

Now we can write down the relations of H as follows:

$TZYX = 1,$

$X^3T^3Z^3Y^3 = 1,$

$X^7T^7Z^7Y^7 = 1,$

$X^{15}T^{15}Z^{15}Y^{15} = 1,$

$X^{31}T^{31}Z^{31}Y^{31} = 1,$

$X^{63}T^{63}Z^{63}Y^{63} = 1,$

"

$X^{(2^{n/2}-1)}T^{(2^{n/2}-1)}Z^{(2^{n/2}-1)}Y^{(2^{n/2}-1)} = 1.$

$X^{2^{(n-2)/2}}T^{(2^{n/2}-1)}Z^{-(2^{(n-2)/2}-1)}T^{(2^{n/2}-1)} = 1,$

$Y^{2^{(n-2)/2}}X^{(2^{n/2}-1)}T^{-(2^{(n-2)/2}-1)}X^{(2^{n/2}-1)} = 1,$

$Z^{2^{(n-2)/2}}Y^{(2^{n/2}-1)}X^{-(2^{(n-2)/2}-1)}Y^{(2^{n/2}-1)} = 1,$

$T^{2^{(n-2)/2}}Z^{(2^{n/2}-1)}Y^{-(2^{(n-2)/2}-1)}Z^{(2^{n/2}-1)} = 1,$

$T^{(2^{n/2}+2^{(n-2)/2}-1)}Z^{(2^{n/2}-1)}Y^{-(2^{(n-2)/2}-1)} = 1,$

$X^{(2^{n/2}+2^{(n-2)/2}-1)}T^{(2^{n/2}-1)}Z^{-(2^{(n-2)/2}-1)} = 1,$

$Y^{(2^{n/2}+2^{(n-2)/2}-1)}X^{(2^{n/2}-1)}T^{-(2^{(n-2)/2}-1)} = 1,$

$Z^{(2^{n/2}+2^{(n-2)/2}-1)}Y^{(2^{n/2}-1)}X^{-(2^{(n-2)/2}-1)} = 1.$

First of all we shall show that H is the derived subgroup of F. Abelianizing the relations of F, from relation (n+4)/2 we can see that $A^{(4.2^{n/2}-1)}B^{(4.2^{n/2}-1)} = 1$ and using relation 1 and 2 we get $A = B^{-1}$. Therefore $|F/F'| = 4$. On the other hand the generators of H are in F' and since the index of H in F is 4, $H = F'$. Now we can show that $F' = C_{(2^{n/2}-1)} \times C_{(2^{n/2}-1)}$.

From relation (n+16)/2 i.e. $Z^{(2^{n/2}+2^{(n-2)/2}-1)}Y^{(2^{n/2}-1)}X^{-(2^{(n-2)/2}-1)} = 1,$

$$X^{-(2^{(n-2)/2}-1)} = Y^{-(2^{n/2}-1)}Z^{-(2^{n/2}+2^{(n-2)/2}-1)} \quad \text{.................................(i).}$$

From relation (n+14)/2 i.e. $Y^{(2^{n/2}+2^{(n-2)/2}-1)}X^{(2^{n/2}-1)}T^{-(2^{(n-2)/2}-1)} = 1,$

$$T^{-(2^{(n-2)/2}-1)} = X^{-(2^{n/2}-1)}Y^{-(2^{n/2}+2^{(n-2)/2}-1)} \quad \text{.................................(ii).}$$

From relation (n+12)/2 i.e. $X^{(2^{n/2}+2^{(n-2)/2}-1)}T^{(2^{n/2}-1)}Z^{-(2^{(n-2)/2}-1)} = 1$,

$Z^{-(2^{(n-2)/2}-1)} = T^{-(2^{n/2}-1)}X^{-(2^{n/2}+2^{(n-2)/2}-1)}$ ....(iii).

From relation (n+10)/2 i.e. $T^{(2^{n/2}+2^{(n-2)/2}-1)}Z^{(2^{n/2}-1)}Y^{-(2^{(n-2)/2}-1)} = 1$,

$Y^{-(2^{(n-2)/2}-1)} = Z^{-(2^{n/2}-1)}T^{-(2^{n/2}+2^{(n-2)/2}-1)}$ ....(iv)

Consider relation (n+8)/2 i.e. $T^{2^{(n-2)/2}}Z^{(2^{n/2}-1)}Y^{-(2^{(n-2)/2}-1)}Z^{(2^{n/2}-1)} = 1$.

Substituting (iv) in this relation we get $T^{(2^{n/2}-1)} = Z^{(2^{n/2}-1)}$ .... (v).

Consider relation (n+6)/2 i.e. $Z^{2^{(n-2)/2}}Y^{(2^{n/2}-1)}X^{-(2^{(n-2)/2}-1)}Y^{(2^{n/2}-1)} = 1$.

Substituting (i) in this relation we get $Y^{(2^{n/2}-1)} = Z^{(2^{n/2}-1)}$ .... (vi).

Consider relation (n+4)/2 i.e. $Y^{2^{(n-2)/2}}X^{(2^{n/2}-1)}T^{-(2^{(n-2)/2}-1)}X^{(2^{n/2}-1)} = 1$.

Substituting (ii) in this relation we get $X^{(2^{n/2}-1)} = Y^{(2^{n/2}-1)}$ .... (vii).

From (v), (vi) and (vii) we get

$T^{(2^{n/2}-1)} = Z^{(2^{n/2}-1)} = T^{(2^{n/2}-1)} = Z^{(2^{n/2}-1)}$ .... (viii).

Now considering relation (n+10)/2 i.e.

$T^{(2^{n/2}+2^{(n-2)/2}-1)}Z^{(2^{n/2}-1)}Y^{-(2^{(n-2)/2}-1)} = 1$.

Using (viii) as $Z^{(2^{n/2}-1)} = T^{(2^{n/2}-1)}$

we get $Y^{-(2^{(n-2)/2}-1)} = T^{-(2.2^{n/2}+2^{(n-2)/2}-2)}$. Squaring both sides we get

$Y^{-2(2^{n/2}-1)}Y = T^{-2(2.2^{n/2}+2^{(n-2)/2}-2)}$.

Using (viii) as $Y^{(2^{n/2}-1)} = T^{(2^{n/2}-1)}$,

$Y = T^{(-2^{(n+4)/2}+3)}$ .... (ix).

Now considering relation (n+16)/2 i.e.

$Z^{(2^{n/2}+2^{(n-2)/2}-1)}Y^{(2^{n/2}-1)}X^{-(2^{(n-2)/2}-1)} = 1$. Using (viii) as $Y^{(2^{n/2}-1)} = Z^{(2^{n/2}-1)}$ we get $X^{-(2^{(n-2)/2}-1)} = Z^{-(2.2^{n/2}+2^{(n-2)/2}-2)}$. Squaring both sides we get $X^{-2(2^{n/2}-1)}Y = Z^{-2(2.2^{n/2}+2^{(n-2)/2}-2)}$. Using (viii) as $X^{(2^{n/2}-1)} = Z^{(2^{n/2}-1)}$ we get $X = Z^{(-2^{(n+4)/2}+3)}$ .... (x).

Now considering relation n/2 i.e.
$X^{(2^{n/2}-1)}T^{(2^{n/2}-1)}Z^{(2^{n/2}-1)}Y^{(2^{n/2}-1)} = 1$. Using (viii) we get $T^{4(2^{n/2}-1)} = 1$ .... (xi).

Now considering relation (n+4)/2 i.e.
$Y^{2^{(n-2)/2}}X^{(2^{n/2}-1)}T^{-(2^{(n-2)/2}-1)}X^{(2^{n/2}-1)} = 1$. Using (viii) as $X^{(2^{n/2}-1)} = T^{(2^{n/2}-1)}$ we can get $Y^{2^{(n-2)/2}}T^{(2.2^{n/2}-2^{(n-2)/2}-1)} = 1$. Using (ix) we get $T^{(2^{n/2}-1)(-2^{n/2+1}+1)} = 1$ ..... (xii).

Now consider relation (n+8)/2 i.e.
$T^{(2^{n/2}-1)}Z^{(2^{n/2}-1)}Y^{(2^{n/2}-1)}Z^{(2^{n/2}-1)} = 1$. Using (viii) as $Z^{(2^{n/2}-1)} = T^{(2^{n/2}-1)}$ we can get $T^{(2.2^{n/2}+2^{(n-2)/2}-2)}Y^{-(2^{(n-2)/2}-1)} = 1$. Using (ix) we get $T^{(2^{n/2}-1)(2^{n/2+1}-1)} = 1$ ..... (xiii).

Now consider relation (n+10)/2 i.e.
$T^{(2^{n/2}+2^{(n-2)/2}-1)}Z^{(2^{n/2}-1)}Y^{-(2^{(n-2)/2}-1)} = 1$. Using (viii) as $Z^{(2^{n/2}-1)} = T^{(2^{n/2}-1)}$ we can get $T^{(2.2^{n/2}+2^{(n-2)/2}-2)}Y^{-(2^{(n-2)/2}-1)} = 1$. Using (ix) we get $T^{(2^{n/2}-1)(2^{n/2+1}-1)} = 1$ ..... (xiv).

Now consider relation (n+12)/2 i.e.
$X^{(2^{n/2}+2^{(n-2)/2}-1)}T^{(2^{n/2}-1)}Z^{-(2^{(n-2)/2}-1)} = 1$. Using (x) and using (viii) as $Z^{(2^{n/2}-1)} = T^{(2^{n/2}-1)}$ we get $T^{(2^{n/2}-1)(-2^{n/2+2}-2^{n/2+1}+3)} = 1$ ..... (xv).

Now consider relation (n+14)/2 i.e.

$Y^{(2^{n/2}+2^{(n-2)/2}-1)}X^{(2^{n/2}-1)}T^{-(2^{(n-2)/2}-1)} = 1$. Using (viii) as $X^{(2^{n/2}-1)} = T^{(2^{n/2}-1)}$ we can get $Y^{(2^{n/2}+2^{(n-2)/2}-1)}T^{(-2^{(n-2)/2}+2^{n/2})} = 1$. Using (ix) we get $T^{(2^{n/2}-1)(-2^{n/2+2}-2^{n/2+1}+3)} = 1$ ..... (xvi).

Now consider relation (n+16)/2 i.e.

$Z^{(2^{n/2}+2^{(n-2)/2}-1)}Y^{(2^{n/2}-1)}X^{-(2^{(n-2)/2}-1)} = 1$. Using (viii) as $Y^{(2^{n/2}-1)} = T^{(2^{n/2}-1)}$ and using (x) we get $T^{(2^{n/2}-1)(+2^{n/2+2}+2^{n/2+1}-3)} = 1$ ..... (xvii).

From (xi), (xii), (xiii), (xiv), (xv), (xvi) and (xvii) we get $T^{(2^{n/2}-1)} = 1$...(xviii)

Now considering (viii) and using (xviii) we get

$T^{(2^{n/2}-1)} = Z^{(2^{n/2}-1)} = T^{(2^{n/2}-1)} = Z^{(2^{n/2}-1)} = 1$ .... (xix).

Now considering relation 1 i.e. TZYX = 1 and using (x) we get

$TZYZ^{(-2^{(n+4)/2}+3)} = 1$. Using (ix) we get

$TZT^{(-2^{(n+4)/2}+3)}Z^{(-2^{(n+4)/2}+3)} = 1$........ (xx).

Using (xix) as $T^{2^{n/2}} = T$, $Z^{2^{n/2}} = Z$ in (xx) we get $TZ = ZT$ so F' is abelian since $T^{(2^{n/2}-1)} = Z^{(2^{n/2}-1)} = 1$.

F' is a direct product of $C_{(2^{n/2}-1)} \times C_{(2^{n/2}-1)}$ because the relations from 2 to n/2 are redundant. Then the order of F' is $(2^{n/2}-1)^2$.

Proof: (ii) From (i) the index of F' in F is 4 and order of F' is $(2^{n/2}-1)^2$. Therefore the order of F is $4.(2^{n/2}-1)^2$.

CHAPTER 3

ON (4,n) – GROUPS

3.0 INTRODUCTION:

We shall be interested in the groups defined by presentations of the form:

$$\langle A,B \mid A^4 = 1, B^n = 1, w(A,B) = 1 \rangle$$

where $n > 1$, $w(A,B)$ is a word in A, B and the exponent sum of B in $w(A,B)$ is zero. Such cases when the exponent sum of B in $w(A,B)$ is zero and $A^2 = 1$ have been considered by Campbell & Thomas, [10] and Doostie, [19]. If $A^3 = 1$, such cases have been considered by David Gill, [20].

In the first section of this chapter we study the groups with a presentation of the form:

$$\langle A,B \mid A^4 = 1, B^n = 1, A^iB^jA^kB^t = 1$$

and determine all possibilities with conditions: $j+t = 0$ and $i,k \in \{ \mp 1, 2 \}$.

In the second section of this chapter we study the groups with a presentation of the form:

$$\langle A,B \mid A^4 = 1, B^n = 1, A^iB^jA^kB^tA^mB^p = 1 \rangle$$

and determine some of the possibilities with conditions:

$j = 1, t = 1, p = -2$ and $i,k,m \in \mathbb{Z}$.

3.1 THE GROUPS G(n; i, j, k, t) :

We shall be interested in groups defined by presentations of the form:

$$\langle A,B \mid A^4 = 1, B^n = 1, w(A,B) = 1 \rangle$$

where $n > 1$, $w(A,B)$ is a word in A, B and the exponent sum of B in $w(A,B)$ is zero. Such a case would be:

$$\langle A,B \mid A^4 = 1, B^n = 1, A^iB^jA^kB^t = 1 \rangle$$

with $j+t = 0$ and $i,k \in \{ \mp 1, 2 \}$.

We can reduce the problem a little by pointing out certain isomorphisms between some of the groups under consideration.

Let $G(n; i, j, k, t)$ denote the group with presentation:

$$\langle A,B \mid A^4 = 1, B^n = 1, A^iB^jA^kB^t = 1 \rangle$$

Lemma 3.1.1 $G(n; i, j, k, t)$ is isomorphic to $G(n; k, t, i, j)$.

Proof: This is clear, since the relation

$$A^iB^jA^kB^t = 1$$

is equivalent to :

$$A^kB^tA^iB^j = 1.$$

Lemma 3.1.2 $G(n; i, j, k, t)$ is isomorphic to $G(n; i, t, k, j)$.

Proof: The relation

$$A^iB^jA^kB^t = 1$$

is equivalent to:

$$A^{-i}B^{-t}A^{-k}B^{-j} = 1.$$

Replacing A by A^{-1} and B by B^{-1} yields the result.

Lemma 3.1.3 $G(n; i, j, k, t)$ is isomorphic to $G(n; -i, j, -k, t)$.

Proof: The relation

$$A^i B^j A^k B^t = 1$$

is equivalent to

$$A^{-i} B^j A^{-k} B^t = 1.$$

Replacing A by A^{-1} yields the result.

Lemma 3.1.4 (1) $G(n; 2, j, k, t) \cong G(n; 2, j, -k, t)$,

(2) $G(n; i, j, 2, t) \cong G(n; -i, j, 2, t)$.

Proof: (1): Use Lemma 3.1.3 and relation 1.

(2): Use Lemma 3.1.3 and relation 1.

Now in view of Lemmas 3.1.1 - 3.1.4 , we have four possible cases to consider and they are :

(1) $G(n; 2, k, 2, -k) \cong G(n; 2, -k, 2, k)$,

(2) $G(n; 1, k, -1, -k) \cong G(n; -1, -k, 1, k) \cong G(n; -1, k, 1, -k) \cong G(n; 1, -k, -1, k)$,

(3) $G(n; 1, k, 1, -k) \cong G(n; 1, -k, 1, k) \cong G(n; -1, k, -1, -k) \cong G(n; -1, -k, -1, k)$,

(4) $G(n; 1, k, 2, -k) \cong G(n; 2, -k, 1, k) \cong G(n; 2, k, 1, -k) \cong G(n; 1, -k, 2, k) \cong$
$G(n; 2, k, -1, -k) \cong G(n; -1, -k, 2, k) \cong G(n; -1, k, 2, -k) \cong G(n; 2, -k, -1, k)$.

Theorem 3.1.1 The group $G(n; 2, k, 2, -k)$ is infinite, for every n and k.

Proof: Adding the relation $A^2 = 1$ to :

$$\langle A, B \mid A^4 = 1, B^n = 1, A^2 B^k A^2 B^{-k} = 1 \rangle$$

we get :

$$\langle A,B \mid A^2 = 1, B^n = 1 \rangle$$

which is a presentation for the infinite group $\mathbb{Z}_2 * \mathbb{Z}_n$.

Theorem 3.1.2

(i) $G(n; 1, k, -1, -k)$ is infinite, if $(n,k) = d$, $d>1$.

(ii) $G(n; 1, k, -1, -k) \cong \mathbb{Z}_4 \times \mathbb{Z}_n$, if $(n,k) = 1$ and $n \equiv 0 (\mathrm{mod} 4)$.

(iii) $G(n; 1,k,-1,-k) \cong \mathbb{Z}_2 \times \mathbb{Z}_n$, if $(n,k) = 1$ and $n \equiv 2 (\mathrm{mod} 4)$.

(iv) $G(n; 1, k, -1, -k) \cong \mathbb{Z}_{4n}$, if $(n,k) = 1$ and n is an odd positive integer.

Proof: (i); Let $(n,k) = d$. Adding the relation $B^d = 1$ to:

$$\langle A,B \mid A^4 = 1, B^n = 1, AB^kA^{-1}B^{-k} = 1 \rangle$$

we get :

$$\langle A,B \mid A^4 = 1, B^d = 1 \rangle$$

which is a presentation for the infinite group $\mathbb{Z}_4 * \mathbb{Z}_d$.

In **(ii)**, **(iii)** and **(iv)**, the relation

$$AB^kA^{-1}B^{-k} = 1$$

yields

$$AB^k = B^kA \text{ .... (I).}$$

If $(n,k) = 1$ then using (I) we see that the group is abelian.

Now, if n is odd then 4n is the only invariant factor of the relation matrix of the group. Therefore $F \cong \mathbb{Z}_{4n}$ and this is the proof for **(iv)**.

If $n \equiv 0 (\mathrm{mod} 4)$ then 4 and n are the only invariant factors of the relation matrix of the group. So $F \cong \mathbb{Z}_4 \times \mathbb{Z}_n$ and this is the proof for **(ii)**.

If $n \equiv 2 (\mathrm{mod} 4)$ then 2 and 2n are the only invariant factors of the relation matrix of the group. So $F \cong \mathbb{Z}_2 \times \mathbb{Z}_{2n}$ and this is the proof for **(iii)**.

Theorem 3.1.3

(i) $G(n; 1, k, 1, -k)$ is infinite, if $(n,k) \neq 1$.

(ii) $G(n; 1, k, 1, -k) \cong \mathbb{Z}_{2n}$, if $(n,k) = 1$ and n is an odd positive integer.

(iii) $G(n; 1, k, 1, -k)$ is a metabelian group of order 4n, if $(n,k) = 1$ and n is an even positive integer.

Proof: (i); Let $(n,k) = d$. Adding the relations $A^2 = 1$, $B^d = 1$, we get a homomorphic image with the presentation :

$$\langle A,B \mid A^2 = 1, B^d = 1 \rangle$$

which is a presentation for the infinite group $Z_2 * Z_d$.

Now we shall give a proof for the other two cases.

Let $G(n; 1, k, 1, -k) = F$ and let $H = \langle A^2, B, ABA^{-1} \rangle < F$. We can define cosets $H = 1$, $1A = 2$. No collapses occur. So the index of H in F is 2. Let us say $X = A^2$, $Y = B$, $Z = ABA^{-1}$ and using R.S. algorithm we can get the following presentation for H:

$$\langle X, Y, Z \mid X^2 = 1, Y^n = 1, Z^n = 1, Z^kXY^{-k} = 1, XY^kZ^{-k} = 1 \rangle.$$

Eliminating the generator X we get:

$$H \cong \langle Y, Z \mid (Z^{-k}Y^k)^2 = 1, Y^n = 1, Z^n = 1, Y^{2k} = Z^{2k} \rangle.$$

From relation 4 we get $Y^k = Z^{2k}Y^{-k}$ .... (*). Using (*) in relation 1 as $Z^{-k}Z^{2k}Y^{-k}Z^{-k}Y^k = 1$ we get

$$Y^kZ^k = Z^kY^k \text{ .... (I).}$$

If $(n,k) = 1$ then $\exists$ f, g $\in \mathbb{Z}$ such that $wn + qk = 1$. Using relation 1 we can get

$$Y^{qk} = Y \text{ .... (II).}$$

Multiplying both sides from the right by Z^k and using (I) we get

$$Z^kY^{qk} = YZ^k \text{ .... (III).}$$

Multiplying both sides of (II) from the left by Z^k we get

$$Z^kY^{qk} = Z^kY \text{ .... (IV).}$$

From (III) and (IV) we get

$$Z^kY = YZ^k \text{ .... (V).}$$

Using wn + qk = 1 and relation 1 we get

$$Z^{qk} = Z \text{ .... (VI).}$$

Multiplying both sides of (VI) from the left by Y we get

$$YZ^{qk} = YZ \text{ .... (VII).}$$

Multiplying both sides from the right of (VI) by Y and using (V) we get

$$YZ^{qk} = ZY \text{ .... (VIII).}$$

From (VII) and (VIII) we get

$$YZ = ZY$$

so Y and Z commute. This means the group is abelian. Therefore its order is given by invariant factors of the relation matrix

$$M = \begin{bmatrix} n & 0 \\ 0 & n \\ 2k & -2k \end{bmatrix}$$

If n is odd then n is the only invariant factor of M and therefore $H \cong \mathbb{Z}_n$. The index of H in F is 2 and so the order of F is 2n. But on the other hand $|F/F'| = 2n$ and so we can derive that F is an abelian group. So its order is given by invariant factors of the relation matrix

$$N = \begin{bmatrix} 2 & 0 \\ 0 & n \end{bmatrix}$$

2n is the only invariant factor of N , n odd, and so $F \cong \mathbb{Z}_{2n}$. This is the proof for **(ii)**.

If n is even then 2 and n are the invariant factors of M. So $H \cong \mathbb{Z}_2 \times \mathbb{Z}_n$. The index of H is 2 and therefore $|F| = 4n$. But on the other hand $|F/F'| = 2n$. This means $|F'| = 2$ and we can say $F' \cong \mathbb{Z}_2$. Then F must be a metabelian group. This is the proof for **(iii)**.

Theorem 3.1.4 For every n and k the group G(n; 1, k , 2, -k) is a cyclic group of order n.

Proof: The relation 3 i.e, $AB^kA^2B^{-k} = 1$, can be rewritten as $A^2 = B^{-k}A^{-1}B^k$. Squaring both sides we get $A^2 = 1$ .... (i). Using (i) in relation 3 we get A = 1. This yields the result.

3.2 THE GROUPS G(n; i, j, k, t, m, p) :

In this section of the chapter we study the groups with a presentation of the form

$$\langle A,B \mid A^4 = 1, B^n = 1, A^iB^jA^kB^tA^mB^p = 1 \rangle$$

and determine some of the possibilities with conditions:

j = 1, t = 1, p = -2 and i,k,m $\in \mathbb{Z}$.

We can reduce the problem a little by pointing out certain isomorphisms between some of the groups under consideration.

Let G(n; i , j , k , t , m , p) denote the group with the presentation:

$$\langle A,B \mid A^4 = 1, B^n = 1, A^iB^jA^kB^tA^mB^p = 1 \rangle$$

Lemma 3.2.1 G(n; i , j, k , t , m , p) is isomorphic to G(n; m , p , i , j , k , t).

Proof: This is clear, since the relation:

$$A^iB^jA^kB^tA^mB^p = 1$$

is equivalent to:

$$A^mB^pA^iB^jA^kB^t = 1.$$

Lemma 3.2.2 G(n; i , j, k , t , m , p) is isomorphic to G(n; i , p, m , t , k , j).

Proof: The relation

$$A^iB^jA^kB^tA^mB^p = 1$$

is equivalent to:

$$A^{-i}B^{-p}A^{-m}B^{-t}A^{-k}B^{-j} = 1.$$

Replacing A by A^{-1} and B by B^{-1} yields the result.

Lemma 3.2.3 G(n; i,j,k,t,m,p) is isomorphic to G(n; -i,j,-k,t,-m,p).

Proof: The relation

$$A^iB^jA^kB^tA^mB^p = 1$$

is equivalent to:

$$A^{-i}B^jA^{-k}B^tA^{-m}B^p = 1.$$

This may be seen by replacing A by A^{-1}.

Corollary 3.2.1 G(n; 2 , j , k , t , m , p) is isomorphic to G(n; 2, j , -k , t ,-m , p).

Proof: Use relation 1 and Lemma 3.2.3 .

Corollary 3.2.2 G(n; 2, j , 2 , t , m , p) is isomorphic to G(n; 2, j , 2 , t , -m , p).

Proof: Use relation 1 and Lemma 3.2.3

Lemma 3.2.4 G(n; i,j,k,t,m,p) is isomorphic to G(n; -m,t,-k,j, -i,p).

Proof: The relation

$$A^iB^jA^kB^tA^mB^p = 1$$

is equivalent to:

$A^{-m}B^{t}A^{-k}B^{j}A^{-i}B^{p} = 1.$

Taking inverse of $A^{i}B^{j}A^{k}B^{t}A^{m}B^{p} = 1$ we get $B^{-p}A^{-m}B^{-t}A^{-k}B^{-j}A^{-i} = 1$, but this is equivalent to $A^{-m}B^{-t}A^{-k}B^{-j}A^{-i}B^{-p} = 1$, replacing B by B^{-1} yields the result.

Corollary 3.2.3 G(n; i,j,k,t,m,p) is isomorphic to G(n; -m,t,-k,j, -i,p).

Proof: Replacing A by A^{-1} and using Lemma 3.2.5 yields the result.

Lemma 3.2.5 **(i)** G(n; 2,j,k,t,m,p) $\cong$ G(n; -2,j,k,t,m,p)

(ii) G(n; i,j,2,t,m,p) $\cong$ G(n; i,j,-2,t,m,p)

(iii) G(n; i,j,k,t,2,p) $\cong$ G(n; i,j,k,t,-2,p)

Proof: **(i)** From relation 1 we get $A^2 = A^{-2}$. Therefore yields the result.

Proof for **(ii)** and **(iii)** is clear from the same argument as in **(i)**.

In view of Lemmas 3.2.1 - 3.2.5 and Corollaries 3.2.1 - 3.2.3 we have ten possible cases to consider they are as follows :

(1) G(n; 1, 1, 1, 1, 1,-2)

(2) G(n; 1, 1, 1, 1, 2, -2)

(3) G(n; 1, 1, 1, 1, -1, -2)

(4) G(n; 1, 1, -1, 1, 1, -2)

(5) G(n; 1, 1, -1, 1, 2, -2)

(6) G(n; -1, 1, 2, 1, 2, -2)

(7) G(n; 2, 1, 2, 1, 2, -2)

(8) G(n; 1, 1, 2, 1, 1, -2)

(9) G(n; 2, 1 , 1, 1, 2, -2)

(10) G(n; 1, 1, 2, 1, -1, -2)

In this section we shall deal with the cases (1), (2), (6), (7), (8) and (10).

Lemma 3.2.6 Let $f_0 = 0$, $f_i = -4f_{i-1} + 2$, $i = 1, 2, 3, \ldots$ then $f_n = (2/5 - 2/5(-4)^n)$.

Proof: We can use induction. For n = 0 the equation is true. Suppose for n = k the following equation

$$f_k = (2/5 - 2/5(-4)^k)$$

to be true.

For $n = k + 1$; $f_{k+1} \quad = -4f_k + 2$

$$= -4(2/5 - 2/5(-4)^k) + 2 \quad \text{by the definition of } f_i$$

$$= ((2/5 - 2/5(-4)^{k+1}).$$

So by induction for every n >1 the claimed equation is true.

CASE (2):

Theorem 3.2.1 Let $F = G(n; 1, 1, 1, 1, 2, -2) =$

$$\langle A,B \mid A^4 = 1, B^n = 1, ABABA^2B^{-2} = 1 \rangle.$$

F has $H = \langle B, ABA^{-1}, A^2 \rangle$ as a subgroup of index 2.

Proof: We can define cosets H = 1, 1A = 2. No collapses occur. Let us put $a = A^2$, $b = B$, $c = ABA^{-1}$. Using the R.S. algorithm we get the following presentation for H.

$$\langle a, b, c \mid a^2 = 1, b^n = 1, c^n = 1, cabab^{-2} = 1, abcac^{-2} = 1 \rangle.$$

Theorem 3.2.2 Let H be the group as in Theorem 3.2.1. For every n the derived group of the group H is of index 2n in H and H' can be presented by n-1 generators $x_0, x_1, x_3, \dots, x_{n-2}$ and 2n-2 relations

$x_{i+2}\, x_{i-1}^{-2}\, x_{i+1} = 1, \quad i = 1, 2, 3, \dots, n-4$

$x_{n-4}^{-2}\, x_{n-2} = 1,$

$x_2 x_1 = 1,$

$x_1\, x_{n-2}^{-2}\, x_0 = 1,$

$x_0\, x_{n-3}^{-2} = 1,$

$[x_i , x_{i-1}] = 1, \qquad i = 1, 2, 3, \dots, n-2$

Proof: Abelianizing the relations of H, the presentation for the factor group will be $\langle a, b, c \mid a^2 = 1, b^n = 1, c^n = 1, cb^{-1} = 1, bc^{-1} = 1, [a,b] = 1, [a,c] = 1, [b,c] = 1 \rangle$. Eliminating the generator c we obtain $\langle a,b \mid a^2 = 1, b^n = 1, [a,b] = 1 \rangle$. So the index of H' in H is 2n.

We can take $U = \{ a^i b^j \mid i = 0,1 \ ; \ j = 0, 1, 2, \ldots, n-1 \}$ as a Schreier transversal for H' in H. Then $\{ ux\underline{ux}^{-1} \mid u \in U, x \in \{a,b\} \}$ generates H' whenever $\underline{ux}$ is an element of U and $ux\underline{ux}^{-1}$ belongs to H' and $\{ uru^{-1} \mid u \in U, \ r \in \{ a^2, b^n, c^n, cabab^{-2}, abcac^{-2} \}\}$ is the set of relations . So we will get

<u>Generators of H':</u>

$x_i = b^{i+1} ab^{-i-1} a^{-1}$, $i = 0, 1, 2, \ldots, n-2$

$x_{n-1} = a^2$

$x_{n+i} = ab^{i+1} ab^{-i-1}$, $i = 0, 1, 2, \ldots, n-2$

$x_{2n-1} = b^n$

$x_{2n} = ab^n a^{-1}$

$x_{2n+i+1} = b^i cb^{-i-1}$, $i = 0, 1, 2, \ldots, n-2$

$x_{3n+i} = ab^i cb^{-i-1} a^{-1}$, $i = 0, 1, 2, \ldots, n-2$

$x_{4n} = ab^{n-1} ca^{-1}$

$x_{4n-1} = b^{n-1} c$

<u>Relations of H' :</u>

$R_1 = a^2, R_2 = ba^2b^{-1}, \ldots, R_n = b^{n-1} a^2 b^{1-n}$,

$R_{n+1} = aba^2b^{-1}a^{-1}, \ldots, R_{2n-1} = ab^{n-1}a^2b^{1-n}a^{-1}$,

$R_{2n} = b^n, R_{2n+1} = ab^n a^{-1}$,

$R_{2n+2} = c^n, R_{2n+3} = bc^n b^{-1}, \ldots, R_{3n+1} = b^{n-1}c^n b^{1-n}$,

$R_{3n+2} = ac^n a^{-1}$,

$R_{3n+3} = abc^n b^{-1}a^{-1}, \ldots, R_{4n+1} = ab^{n-1}c^n b^{1-n}a^{-1}$,

$R_{4n+2} = cabab^{-2}, R_{4n+3} = b(cabab^{-2})b^{-1}, \ldots, R_{5n+1} = b^{n-1}(cabab^{-2})b^{1-n}$,

$R_{5n+2} = ab(cabab^{-2})b^{-1}a^{-1}, \dots, R_{6n} = ab^{n-1}(cabab^{-2})b^{1-n}a^{-1},$

$R_{6n+1} = a(cabab^{-2})a^{-1},$

$R_{6n+2} = abcac^{-2}, R_{6n+3} = b(abcac^{-2})b^{-1}, \dots, R_{7n+1} = b^{n-1}(abcac^{-2})b^{1-n},$

$R_{7n+2} = ab(abcac^{-2})b^{-1}a^{-1}, \dots, R_{8n} = ab^{n-1}(abcac^{-2})b^{1-n}a^{-1},$

$R_{8n+1} = a(abcac^{-2})a^{-1}$

The presentation of H' is as follows:

$$H' = \langle x_0, x_1, x_3, \dots, x_{4n} \mid R_i = 1,\ i = 1,2, \dots, 8n+1 \rangle$$

Now we have to describe R_i in terms of x_i. We see that

$$x_{n-1} = x_{2n-1} = x_{2n} = 1$$

$R_1 = x_{n-1}$

$R_{i+2} = x_i x_{i+n} \quad , i = 0,1,2, \dots, n-2$

$R_{n+1+i} = x_{i+n} x_i \qquad , i = 0,1,2, \dots, n-2$

$R_{2n} = x_{2n-1}$

$R_{2n+1} = x_{2n}$

$R_{2n+2} = \dots = R_{3n+1} = \left(\prod_{i=1}^{n-1} x_{2n+i}\right)x_{4n-1}$

$R_{3n+2} = \dots = R_{4n+1} = \left(\prod_{i=0}^{n-2} x_{3n+i}\right)x_{4n}$

We notice that for every i, i = 2n+3, ... , 3n+1 the relations $R_i = 1$ are redundant from the relation $R_{2n+2} = 1$. For every i, i = 3n+3, ... , 4n+1 the relations $R_i = 1$ are redundant from the relation $R_{3n+2} = 1$. For every i, i = 0, 1, ... , n-2 the relations $R_{n+1+i} = 1$ are redundant from the relations $R_{i+2} = 1$, i = 0, 1, ... , n-2.

Consider foregoing relations. Then we will get :

$R_{4n+2} = x_{2n+1}\, x_0\, x_{n+1}$, $R_{4n+3} = x_{2n+1}\, x_1\, x_{n+2}$, ... , $R_{5n-1} = x_{3n-2}\, x_{n-3}\, x_{2n-2}$

$R_{5n} = x_{3n-1}\, x_{n-2}$,

$R_{5n+1} = x_{4n-1}\, x_n$,

$R_{5n+2} = x_{3n+1}\, x_{n+1}\, x_2$, $R_{5n+3} = x_{3n+2}\, x_{n+2}\, x_3$, ... , $R_{6n-2} = x_{4n-3}\, x_{2n-3}\, x_{n-2}$,

$R_{6n-1} = x_{4n-2}\, x_{2n-2}$,

$R_{6n} = x_{4n} x_0$,

$R_{6n+1} = x_{3n} x_n$,

$R_{6n+2} = x_{3n+1} x_{n+1} x_{2n+2}^{-1} x_{2n+1}^{-1}$,

$R_{6n+3} = x_0 x_{3n+2} x_{n+2} x_{2n+3}^{-1} x_{2n+2}^{-1}$,

$R_{6n+4} = x_1 x_{3n+3} x_{n+3} x_{2n+4}^{-1} x_{2n+3}^{-1}$,

"

$R_{7n-1} = x_{n-4} x_{4n-2} x_{2n-2} x_{3n-1}^{-1} x_{3n-2}^{-1}$,

$R_{7n} = x_{n-3} x_{4n} x_{4n-1}^{-1} x_{3n-1}^{-1}$,

$R_{7n+1} = x_{n-2} x_{3n} x_n x_{2n+1}^{-1} x_{4n-1}^{-1}$,

$R_{7n+2} = x_n x_{2n+3} x_2 x_{3n+2}^{-1} x_{3n+1}^{-1}$,

$R_{7n+3} = x_{n+1} x_{2n+4} x_2 x_{3n+2}^{-1} x_{3n+1}^{-1}$,

"

$R_{8n-2} = x_{2n-4} x_{3n-1} x_{n-2} x_{4n-2}^{-1} x_{4n-3}^{-1}$,

$R_{8n-1} = x_{2n-3} x_{4n-1} x_{4n}^{-1} x_{4n-2}^{-1}$,

$R_{8n} = x_{2n-2} x_{2n+1} x_0 x_{3n}^{-1} x_{4n}^{-1}$,

$R_{8n+1} = x_{2n+2} x_1 x_{3n+1}^{-1} x_{3n}^{-1}$,

Using ($R_{i+2} = x_i x_{i+n}$, $i = 0, 1, \dots, n-2$) we can see that $x_{i+n} = x_i^{-1}$, $i = 0, 1, \dots, n-2$. Eliminating the generators x_{n+i} , $i = 0, \dots, n-2$ we can see that the remaining relations are

$R_{2n+2} = (\prod_{i=1}^{n-1} x_{2n+i})x_{4n-1}$

$R_{3n+2} = (\prod_{i=0}^{n-2} x_{3n+i})x_{4n}$

$R_{4n+2} = x_{2n+1} x_0 x_1^{-1}$,

$R_{4n+3} = x_{2n+2} x_1 x_2^{-1}$,

"

$R_{5n-1} = x_{3n-2} x_{n-3} x_{n-2}^{-1}$,

$R_{5n} = x_{3n-1} x_{n-2}$,

$R_{5n+1} = x_{4n-1}\, x_0^{-1}$,

$R_{5n+2} = x_{3n+1}\, x_1^{-1} x_2$,

$R_{5n+3} = x_{3n+2}\, x_2^{-1} x_3$,

$R_{5n+4} = x_{3n+3}\, x_3^{-1} x_4$,

"

$R_{6n-2} = x_{4n-3}\, x_{n-3}^{-1} x_{n-2}$,

$R_{6n-1} = x_{4n-2}\, x_{n-2}^{-1}$,

$R_{6n} = x_{4n}\, x_0$,

$R_{6n+1} = x_{3n}\, x_0^{-1} x_1$,

$R_{6n+2} = x_{3n+1}\, x_1^{-1}\, x_{2n+2}^{-1} x_{2n+1}^{-1}$,

$R_{6n+3} = x_0 x_{3n+2}\, x_2^{-1}\, x_{2n+3}^{-1} x_{2n+2}^{-1}$,

$R_{6n+4} = x_1 x_{3n+3}\, x_2^{-1}\, x_{2n+4}^{-1} x_{2n+3}^{-1}$,

"

$R_{7n-1} = x_{n-4} x_{4n-2}\, x_{n-2}^{-1}\, x_{3n-1}^{-1} x_{3n-2}^{-1}$,

$R_{7n} = x_{n-3}\, x_{4n}\, x_{4n-1}^{-1}\, x_{3n-1}^{-1}$,

$R_{7n+1} = x_{n-2}\, x_{3n}\, x_0^{-1}\, x_{2n+1}^{-1}\, x_{4n-1}^{-1}$,

$R_{7n+2} = x_0^{-1}\, x_{2n+3}\, x_2\, x_{3n+2}^{-1}\, x_{3n+1}^{-1}$,

$R_{7n+3} = x_1^{-1}\, x_{2n+4}\, x_3\, x_{3n+3}^{-1}\, x_{3n+2}^{-1}$,

"

$R_{8n-2} = x_{n-4}^{-1}\, x_{3n-1}\, x_{n-2}\, x_{4n-2}^{-1}\, x_{4n-3}^{-1}$,

$R_{8n-1} = x_{n-3}^{-1}\, x_{4n-1}\, x_{4n}^{-1}\, x_{4n-2}^{-1}$,

$R_{8n} = x_{n-2}^{-1}\, x_{2n+1}\, x_0\, x_{3n}^{-1}\, x_{4n}^{-1}$,

$R_{8n+1} = x_{2n+2}\, x_1\, x_{3n+1}^{-1}\, x_{3n}^{-1}$,

From relation:	we can get:
R_{4n+2}	$x_{2n+1} = x_1\, x_0^{-1}$
R_{4n+3}	$x_{2n+2} = x_2\, x_1^{-1}$
"	"

R_{5n-1}	$x_{3n-2} = x_{n-2}\, x_{n-3}^{-1}$
R_{5n}	$x_{3n-1} = x_{n-2}^{-1}$
R_{5n+1}	$x_{4n-1} = x_0$
R_{5n+2}	$x_{3n+1} = x_2^{-1}\, x_1$
R_{5n+3}	$x_{3n+2} = x_3^{-1}\, x_2$
R_{5n+4}	$x_{3n+3} = x_4^{-1}\, x_3$
"	"
"	"
R_{6n-2}	$x_{4n-3} = x_{n-2}^{-1}\, x_{n-3}$
R_{6n-1}	$x_{4n-2} = x_{n-2}$
R_{6n}	$x_{4n} = x_0^{-1}$
R_{6n+1}	$x_{3n} = x_1^{-1}\, x_0$

Eliminating $x_{2n+1}, \dots, x_{3n-1}, x_{3n}, \dots, x_{4n}$ the remaining relations are as follows :

$R_{2n+2} = (\prod_{i=1}^{n-2} x_i\, x_{i-1}^{-1})x_{n-2}^{-1}\, x_0$

$R_{3n+2} = (\prod_{i=0}^{n-3} x_{i+1}^{-1}\, x_i)x_{n-2}\, x_0^{-1}$

$R_{6n+2} = x_2^{-1}\, x_1\, x_2^{-1} x_0\, x_1^{-1}$,

$R_{6n+3} = x_0\, x_3^{-1}\, x_2 x_3^{-1} x_1\, x_2^{-1}$,

$R_{6n+4} = x_1\, x_4^{-1}\, x_3 x_4^{-1} x_2\, x_3^{-1}$,

"

"

$R_{7n-1} = x_{n-4}\, x_{n-2}\, x_{n-3}\, x_{n-2}^{-1}$,

$R_{7n} = x_{n-3}\, x_0^{-2}\, x_{n-2}$,

$R_{7n+1} = x_{n-2}\, x_1^{-1}\, x_0 x_1^{-1} x_0^{-1}$,

$R_{7n+2} = x_0^{-1} x_3 x_2^{-1}\, x_3 x_1^{-1} x_2$,

$$R_{7n+3} = x_1^{-1}x_4x_3^{-1}x_4x_2^{-1}x_3 \quad ,$$

"

$$R_{8n-2} = x_{n-4}^{-1}x_{n-2}^{-1}x_{n-3}^{-1}x_{n-2} \quad ,$$

$$R_{8n-1} = x_{n-3}^{-1}x_0^{2}x_{n-2}^{-1} \quad ,$$

$$R_{8n} = x_{n-2}^{-1}x_1x_0^{-1}x_1x_0 \quad ,$$

$$R_{8n+1} = x_2x_1^{-1}x_2x_0^{-1}x_1 \quad ,$$

We can renumber the remaining relations as:

$$R_{2n+1} = \left(\prod_{i=1}^{n-2} x_i x_{i-1}^{-1}\right)x_{n-2}^{-1}x_0 \quad ,$$

$$R_{2n+2} = \left(\prod_{i=0}^{n-3} x_{i+1}^{-1}x_i\right)x_{n-2}x_0^{-1} \quad ,$$

$$R_i = x_{i-1}x_{i+2}^{-1}x_{i+1}x_{i+2}^{-1}x_ix_{i+1}^{-1} \quad , \quad i = 1, 2, \ldots, n-4 \quad ,$$

$$R_{n-3} = x_2^{-1}x_1x_2^{-1}x_0x_1^{-1} \quad ,$$

$$R_{n-2} = x_{n-4}x_{n-2}x_{n-3}x_{n-2}^{-1} \quad ,$$

$$R_{n-1} = x_{n-3}x_0^{-2}x_{n-2} \quad ,$$

$$R_n = x_{n-2}x_1^{-1}x_0x_1^{-1}x_0^{-1} \quad ,$$

$$R_{n+i} = x_{i-1}^{-1}x_{i+2}x_{i+1}^{-1}x_{i+2}x_i^{-1}x_{i+1} \quad , \quad i = 1, 2, \ldots, n-4 \quad ,$$

$$R_{2n-3} = x_{n-4}^{-1}x_{n-2}^{-1}x_{n-3}^{-1}x_{n-2} \quad ,$$

$$R_{2n-2} = x_{n-3}^{-1}x_0^{2}x_{n-2}^{-1} \quad ,$$

$$R_{2n-1} = x_{n-2}^{-1}x_1x_0^{-1}x_1x_0 \quad ,$$

$$R_{2n} = x_2x_1^{-1}x_2x_0^{-1}x_1 \quad ,$$

Using relation R_{n-1} and R_{2n-2} we get

$$[x_{n-3}, x_{n-2}] = 1 \text{ .... (i).}$$

Using (i) in relation R_{n-2} we obtain $x_{n-4}x_{n-3} = 1$, so

$$[x_{n-4}, x_{n-3}] = 1 \text{ .... (ii).}$$

Using (i) and (ii) in R_{n-4} we get

$$x_{n-5}x_{n-2}^{-2}x_{n-4} = 1 \text{ .... (iii).}$$

Using (i) and (ii) in R_{2n-4} we obtain

$$x_{n-5}^{-1}\, x_{n-2}^{2}\, x_{n-4}^{-1} = 1 \ldots\ldots\ldots \text{(iv)}.$$

Using (iii) and (iv) we get

$$[x_{n-5}\, ,\, x_{n-4}] = 1 \ldots\ldots\ldots \text{(v)}.$$

Continuing in the same way, by induction we see that for $i = 1, \ldots, n\text{-}2$

$$[x_{i-1}\, ,\, x_i] = 1 \ldots\ldots\ldots \text{(vi)}.$$

Using (vi) we see that the relations R_{2n+1} and R_{2n+2} are trivial relations.

Using (vi) for $i = 1, \ldots, n\text{-}4$ the relations R_i will become:

$$R_i = x_{i-1}\, x_{i+2}^{-2}\, x_i \quad .$$

Using (vi) for $i = 1, \ldots, n\text{-}4$ the relations R_{n+i} become:

$$R_{n+i} = x_{i-1}^{-1}\, x_{i+2}^{2}\, x_i^{-1} \quad .$$

Using (vi) the relations R_{n-3} , R_{n-2} , R_{n-1} , R_n , R_{2n-3} , R_{2n-2} , R_{2n-1} , R_{2n} will become respectively :

$R_{n-3} = x_2^{-2}\, x_0$,

$R_{n-2} = x_{n-4}\, x_{n-3}$,

$R_{n-1} = x_{n-3}\, x_0^{-2}\, x_{n-2}$,

$R_n = x_{n-2}\, x_1^{-2}$,

$R_{2n-3} = x_{n-4}^{-1}\, x_{n-3}^{-1}$,

$R_{2n-2} = x_{n-3}^{-1}\, x_0^{2}\, x_{n-2}^{-1}$,

$R_{2n-1} = x_{n-2}^{-1}\, x_1^{2}$,

$R_{2n} = x_2^{2}\, x_0^{-1}$,

Now we can see that for $i = 1, \ldots, n\text{-}4$, the relations R_{n+i} are redundant using the relations R_i .

R_{2n} is redundant from R_{n-3} ,

R_{2n-1} is redundant from R_n ,

R_{2n-2} is redundant from R_{n-1} ,

R_{2n-3} is redundant from R_{n-2} .

Removing redundant relations we obtain the following relations for H'

$$R_i = x_{i-1}x_{2+i}^{-2}x_i \quad , \quad i = 1, \dots, n-4$$

$$R_{n-3} = x_2^{-2}x_0 \quad ,$$

$$R_{n-2} = x_{n-4}x_{n-3} \quad ,$$

$$R_{n-1} = x_{n-3}x_0^{-2}x_{n-2} \quad ,$$

$$R_n = x_{n-2}x_1^{-2} \quad ,$$

$$R_{n+i} = [x_{i-1}, x_i] = 1 , i = 1, \dots, n-2.$$

We can rename the generators as :

$$x_i \mapsto x_{n-(2+i)} \quad , i = 0, 1, \dots, n-2$$

and replacing them in the above relations the presentation for H' will become as claimed.

Lemma 3.2.7 Let H' be as in Theorem 3.2.2. If $n \geq 8$ and $n \equiv 0 (\mathrm{mod} 4)$ then H' can be generated by x_0 and x_1 and H' has the following presentation:

$$H' = \langle x_0, x_1 \mid x_1^{(-1+(-4)^{n/4})/5}x_0^{2(-1+(-4)^{n/4})/5} = 1,\ x_1^{3(-1+(-4)^{n/4})/5}x_0^{-1(-1+(-4)^{n/4})/5} = 1,$$
$$x_1^{2(-1+(-4)^{n/4})/5}x_0^{-1(-1+(-4)^{n/4})/5} = 1, [x_1, x_0] = 1 \rangle.$$

Proof: From relation R_{n-2} we get

$$x_2 = x_1^{-1}x_0^{0} \text{ .... (i).}$$

From relation R_1, using (i) and $[x_1, x_0] = 1$, we get

$$x_3 = x_1 x_0^{2} \text{ .... (ii).}$$

Using the relations R_i and formulae given in Lemma 3.2.6 for $i = 1, \dots, n-4$ we can see by induction that

for $t = 1, \dots, (n-4)/4$, $\quad x_{4t} = x_1^{(1-(-4)^t)/5}x_0^{(2+3(-4)^t)/5}$ .... (At).

for $t = 1, \dots, (n-4)/4$, $\quad x_{4t+1} = x_1^{(1+4(-4)^t)/5}x_0^{(2-2(-4)^t)/5}$ .... (Bt)

for $t = 1, \dots, (n-4)/4$, $\quad x_{4t+2} = x_1^{(1-6(-4)^t)/5}x_0^{(2-2(-4)^t)/5}$ ....(Ct)

for $t = 1, \dots, (n-8)/4$, $\quad x_{4t+3} = x_1^{(1+4(-4)^t)/5}x_0^{(2+8(-4)^t)/5}$ .... (Dt).

So the generators $x_2, x_3, \dots, x_{n-2}$ can be replaced with their new values in terms of x_1 and x_0. We are going to show that the relations $R_i = 1$, $i = 1, 2, \dots, n-4$ are trivial relations.

We can start from the first relation.

$$R_1 = x_3 x_0^{-2} x_2$$

$$R_1 = x_1 x_0^{2} x_0^{-2} x_1^{-1} \quad \text{, using (i) and (ii)}$$

$$R_1 = x_1^{0} x_0^{0} = 1.$$

Now the second relation is

$$R_2 = x_4 x_1^{-2} x_3.$$

Using (At, t = 1), (ii) and relation R_{n+1}

$$R_2 = x_1^{(1-(-4)^1)/5} x_0^{(2+3(-4)^1)/5} x_1^{-2} x_1 x_0^{2},$$

$$R_2 = x_1^{0} x_0^{0} = 1.$$

Now the 3 rd relation is

$$R_3 = x_5 x_2^{-2} x_4.$$

Using (Bt,t=1), (i), and (At,t=1)

$$R_3 = x_1^{(1+4(-4)^1)/5} x_0^{(2-2(-4)^1)/5} x_1^{2} x_1^{(1-(-4)^1)/5} x_0^{(2+3(-4)^1)/5},$$

using relation R_{n+1}

$$R_3 = x_1^{0} x_0^{0} = 1.$$

Now the fourth relation is

$$R_4 = x_6 x_3^{-2} x_5.$$

Using (Bt, t=1), (ii) and (Ct, t=1)

$$R_4 = x_1^{(1-6(-4)^1)/5} x_0^{(2-2(-4)^1)/5} x_1^{-2} x_0^{-4} x_1^{(1+4(-4)^1)/5} x_0^{(2-2(-4)^1)/5},$$

using relation R_{n+1}

$$R_4 = x_1^{0} x_0^{0} = 1.$$

Now for $R_{4t+1} = x_{4t+3} x_{4t}^{-2} x_{4t+2}$, $t = 1, \dots, (n-8)/4$.

For $t = 1, \dots, (n-8)/4$ using Dt, At and Ct we can get

$$R_{4t+1} = x_1^{(1+4(-4)^t)/5} x_0^{(2+8(-4)^t)/5} \left(x_1^{(1-(-4)^t)/5} x_0^{(2+3(-4)^t)/5} \right)^{-2} x_1^{(1-6(-4)^t)/5} x_0^{(2-2(-4)^t)/5}$$

Using relation R_{n+1}

$R_{4t+1} = x_1^0 x_0^0 = 1.$

Now for $R_{4t+2} = x_{4(t+1)} x_{4t+1}^{-2} x_{4t+3}$, $t = 1, \dots, (n-8)/4$.

For $t = 1, \dots, (n-8)/4$ using At, Bt and Dt we can get

$$R_{4t+2} = x_1^{(1-(-4)^{t+1})/5} x_0^{(2+3(-4)^{t+1})/5} \left(x_1^{(1+4(-4)^t)/5} x_0^{(2-2(-4)^t)/5} \right)^{-2} x_1^{(1+4(-4)^t)/5} x_0^{(2+8(-4)^t)/5}$$

Using relation R_{n+1}

$R_{4t+2} = x_1^0 x_0^0 = 1.$

Now for $R_{4t+3} = x_{4(t+1)+1} x_{4t+2}^{-2} x_{4(t+1)}$, $t = 1, \dots, (n-8)/4$.

For $t = 1, \dots, (n-8)/4$ using Bt, Ct and At we can get

$$R_{4t+3} = x_1^{(1+4(-4)^{t+1})/5} x_0^{(2-2(-4)^{t+1})/5} \left(x_1^{(1-6(-4)^t)/5} x_0^{(2-2(-4)^t)/5} \right)^{-2} x_1^{(1-(-4)^{t+1})/5} x_0^{(2+3(-4)^{t+1})/5}$$

Using relation R_{n+1}

$R_{4t+3} = x_1^0 x_0^0 = 1.$

Now for $R_{4(t+1)} = x_{4(t+1)+2} x_{4t+3}^{-2} x_{4(t+1)+1}$, $t = 1, \dots, (n-8)/4$.

For $t = 1, \dots, (n-8)/4$ using Ct, Dt and Bt we can write $R_{4(t+1)}$ as

$$x_1^{(1-6(-4)^{t+1})/5} x_0^{(2-2(-4)^{t+1})/5} \left(x_1^{(1+4(-4)^t)/5} x_0^{(2+8(-4)^t)/5} \right)^{-2} x_1^{(1+4(-4)^{t+1})/5} x_0^{(2-2(-4)^{t+1})/5}$$

Using relation R_{n+1}

$R_{4(t+1)} = x_1^0 x_0^0 = 1.$

It is obvious that for every i, $i = 2, \dots, n-2$ the relations $R_{n+i} = [x_i, x_{i-1}]$ are trivial relations and the relation R_{n-2} is trivial (This can be seen using (i)). The remaining relations are as:

$$R_{n+1} = [x_1, x_0],$$
$$R_{n-3} = x_1^{(-1+(-4)^{n/4})/5} x_0^{2(-1+(-4)^{n/4})/5} = 1,$$
$$R_{n-1} = x_1^{3(-1+(-4)^{n/4})/5} x_0^{-1(-1+(-4)^{n/4})/5} = 1,$$
$$R_n = x_1^{2(-1+(-4)^{n/4})/5} x_0^{-1(-1+(-4)^{n/4})/5} = 1.$$

So the presentation for H' is as claimed.

Theorem 3.2.3 Let H' be as in Lemma 3.2.7. If $n \geq 8$ and $n \equiv 0(\mathrm{mod}4)$ then $H' = C_A \times C_{5A}$, where $A = (-1+(-4)^{n/4})/5$. The order of H' is $5((-1+(-4)^{n/4})/5)^2$.

Proof: Suppose $n \geq 8$ and $n \equiv 0(\mathrm{mod}4)$. Then by Lemma 3.2.7 H' is an abelian group. We can consider the presentation of H' given in Lemma 3.2.7. Its order can be given by invariant factors of the relation matrix of this presentation. Let $A = (-1+(-4)^{n/4})/5$. Then the relation matrix will be as follows

$$M = \begin{bmatrix} A & 2A \\ -3A & -A \\ 2A & -A \end{bmatrix}.$$

The only invariant factors of this matrix are A and 5A, so H' is a direct product of two cyclic groups of orders $(-1+(-4)^{n/4})/5$ and $(-1+(-4)^{n/4})$ respectively. Therefore the order of H' is $5((-1+(-4)^{n/4})/5)^2$.

Theorem 3.2.4 Let F be as in Theorem 3.2.1. If $n \geq 8$ and $n \equiv 0(\mathrm{mod}4)$ then F has order $20nA^2$, where $A = (-1+(-4)^{n/4})/5$.

Proof: Suppose $n \geq 8$ and $n \equiv 0(\mathrm{mod}4)$. Then by Theorem 3.2.1, F has H as a subgroup of index 2. By Theorem 3.2.2 the derived subgroup of H has index 2n in H. By Theorem 3.2.3 the order of H' is $5((-1+(-4)^{n/4})/5)^2$ and therefore the order of F is $20n((-1+(-4)^{n/4})/5)^2$.

Lemma 3.2.8 Let H' be as in Theorem 3.2.2. If $n \geq 9$ and $n \equiv 1(\mathrm{mod}4)$ then

(i) H' can be generated by x_0 and x_1 and H' has the following presentation:

$$H' = \langle x_0, x_1 \mid x_1^{(-1+(-4)^{(n-1)/4})/5} x_0^{(-2-3(-4)^{(n-1)/4})/5} = 1,$$
$$x_1^{(3+2(-4)^{(n-1)/4})/5} x_0^{(1+4(-4)^{(n-1)/4})/5} = 1,$$
$$x_1^{(-2-3(-4)^{(n-1)/4})/5} x_0^{(1-(-4)^{(n-1)/4})/5} = 1, \ [x_1, x_0] = 1 \quad \rangle.$$

(ii) Its order is given by invariant factors of the relation matrix

$$M_1 = \begin{bmatrix} -1/5 + 1/5A & -2/5 - 3/5A \\ 3/5 + 2/5A & 1/5 + 4/5A \\ -2/5 - 3/5A & 1/5 - 1/5A \end{bmatrix}$$

where $A = (-4)^{(n-1)/4}$, which is $(2((-4)^{(n-1)/4})^2 + 2\,(-4)^{(n-1)/4} + 1)/5$.

Proof: **(i)** Here we are considering the relations for H' given in Theorem 3.2.2 . Using the relation R_{n-2} we get

$$x_2 = x_1^{-1} \;................\; \text{(i)}.$$

From relation R_1 and using (i) and $[x_1, x_0] = 1$ we get

$$x_3 = x_1\, x_0^2 \;................\; \text{(ii)}.$$

Using the relations R_i and formulae given in Lemma 3.2.6, for $i = 1, \dots, n\text{-}4$

we can see by induction that

for $t = 1, \dots, (n\text{-}5)/4$, $\quad x_{4t} = x_1^{(1-(-4)^t)/5}\, x_0^{(2+3(-4)^t)/5}$ .... (At).

for $t = 1, \dots, (n\text{-}5)/4$, $\quad x_{4t+1} = x_1^{(1+4(-4)^t)/5}\, x_0^{(2-2(-4)^t)/5}$ .... (Bt)

for $t = 1, \dots, (n\text{-}5)/4$, $\quad x_{4t+2} = x_1^{(1-6(-4)^t)/5} x_0^{(2-2(-4)^t)/5}$ ....(Ct)

for $t = 1, \dots, (n\text{-}5)/4$, $\quad x_{4t+3} = x_1^{(1+4(-4)^t)/5} x_0^{(2+8(-4)^t)/5}$ .... (Dt).

So the generators $x_2, x_3, \dots, x_{n-2}$ can be replaced with their new values in terms of x_1 and x_0 . As in Lemma 3.2.7 it can be shown that the relations $R_i = 1$, $i = 1, 2, \dots, n\text{-}4$ are trivial relations. It is obvious that for every i, $i = 1,2, \dots n\text{-}2$ the relations $R_{n+i} = [x_i, x_{i-1}]$ are trivial relations. The relation R_{n-2} is trivial which can be shown using (i). The remaining relations are :

$$R_{n+1} = [x_1, x_0],$$
$$R_{n-3} = x_1^{(-1+(-4)^{(n-1)/4})/5} x_0^{(-2-3(-4)^{(n-1)/4})/5} = 1,$$
$$R_{n-1} = x_1^{(3+2(-4)^{(n-1)/4})/5} x_0^{(1+4(-4)^{(n-1)/4})/5} = 1,$$
$$R_n = x_1^{(-2-3(-4)^{(n-1)/4})/5} x_0^{(1-(-4)^{(n-1)/4})/5} = 1,$$

So the presentation for H' will be as claimed.

(ii) From **(i)** we can see that H' is abelian. Therefore its order is given by invariant factors of the relation matrix

$$M_1 = \begin{bmatrix} -1/5 + 1/5A & -2/5 - 3/5A \\ 3/5 + 2/5A & 1/5 + 4/5A \\ -2/5 - 3/5A & 1/5 - 1/5A \end{bmatrix}$$

where A is $(-4)^{(n-1)/4}$. The only invariant factor of M_1 is $(2A^2 + 2A + 1)/5$. So H' is the cyclic group of order $(2((-4)^{(n-1)/4})^2 + 2(-4)^{(n-1)/4} + 1)/5$.

Theorem 3.2.4 Let F be as in Theorem 3.2.1. If $n \geq 9$ and $n \equiv 1 (\text{mod} 4)$ then F has order $4n(2((-4)^{(n-1)/4})^2 + 2(-4)^{(n-1)/4} + 1)/5$.

Proof: Suppose $n \geq 9$ and $n \equiv 1 (\text{mod} 4)$. Then by Theorem 3.2.1, F has H as a subgroup of index 2. By Theorem 3.2.2 the derived subgroup of H has index 2n in H. By Lemma 3.2.8 the order of H' is $(2((-4)^{(n-1)/4})^2 + 2(-4)^{(n-1)/4} + 1)/5$. Therefore the order of F is $4n(2((-4)^{(n-1)/4})^2 + 2(-4)^{(n-1)/4} + 1)/5$.

Lemma 3.2.9 Let H' be as in Theorem 3.2.2. If $n \geq 10$ and $n \equiv 2 (\text{mod} 4)$ then

(i) H' can be generated by x_0 and x_1 and H' has the following presentation:

$$\begin{aligned} H' = \langle x_0, x_1 \mid\ & x_1^{(-1-4(-4)^{(n-2)/4})/5} x_0^{(-2+2(-4)^{(n-2)/4})/5} = 1, \\ & x_1^{(3+2(-4)^{(n-2)/4})/5} x_0^{(1-6(-4)^{(n-2)/4})/5} = 1, \\ & x_1^{(-2+2(-4)^{(n-2)/4})/5} x_0^{(1+4(-4)^{(n-2)/4})/5} = 1, \qquad [x_1, x_0] = 1 \quad \rangle. \end{aligned}$$

(ii) Its order is given by invariant factors of the relation matrix

$$M_2 = \begin{bmatrix} -1/5 - 4/5A & -2/5 + 2/5A \\ 3/5 + 2/5A & 1/5 - 6/5A \\ -2/5 + 2/5A & 1/5 + 4/5A \end{bmatrix}$$

where $A = (-4)^{(n-2)/4}$. This is $(1 + 4((-4)^{(n-2)/4})^2)/5$.

Proof: **(i)** Here we are considering the relations for H' given in Theorem 3.2.2 . Using the relation R_{n-2} we can get

$$x_2 = x_1^{-1} \text{ .... (i).}$$

From relation R_1 and using (i) and $[x_1, x_0] = 1$ we get

$$x_3 = x_1 x_0^2 \text{ .... (ii).}$$

Using the relations R_i and formulae given in Lemma 3.2.6, for $i = 1, \dots, n-4$ we can see by induction that

for $t = 1, \dots, (n-2)/4$, $\quad x_{4t} = x_1^{(1-(-4)^t)/5} x_0^{(2+3(-4)^t)/5}$ .... (At).

for $t = 1, \dots, (n-6)/4$, $\quad x_{4t+1} = x_1^{(1+4(-4)^t)/5} x_0^{(2-2(-4)^t)/5}$ .... (Bt)

for $t = 1, \dots, (n-6)/4$, $\quad x_{4t+2} = x_1^{(1-6(-4)^t)/5} x_0^{(2-2(-4)^t)/5}$ ....(Ct)

for $t = 1, \dots, (n-6)/4$, $\quad x_{4t+3} = x_1^{(1+4(-4)^t)/5} x_0^{(2+8(-4)^t)/5}$ .... (Dt).

So the generators $x_2, x_3, \dots, x_{n-2}$ can be replaced with their new values in terms of x_1 and x_0 . As in Lemma 3.2.7 it can be shown that the relations $R_i = 1$, $i = 1, 2, \dots, n-4$ are the trivial relations. It is obvious that for every i, $i = 1,2, \dots n-2$ the relations $R_{n+i} = [x_i, x_{i-1}]$ are the trivial relations. The relation R_{n-2} is trivial which can be shown using (i). The remaining relations are :

$$R_{n+1} = [x_1, x_0],$$
$$R_{n-3} = x_1^{(-1-4(-4)^{(n-2)/4})/5} x_0^{(-2+2(-4)^{(n-2)/4})/5} = 1,$$
$$R_{n-1} = x_1^{(3+2(-4)^{(n-2)/4})/5} x_0^{(1-6(-4)^{(n-2)/4})/5} = 1,$$
$$R_n = x_1^{(-2+2(-4)^{(n-2)/4})/5} x_0^{(1+4(-4)^{(n-2)/4})/5} = 1,$$

So the presentation for H' will be as claimed.

(ii) From **(i)** we can see that H' is abelian. Therefore its order is given by the invariant factors of the relation matrix

$$M_2 = \begin{bmatrix} -1/5 - 4/5A & -2/5 + 2/5A \\ 3/5 + 2/5A & 1/5 - 6/5A \\ -2/5 + 2/5A & 1/5 + 4/5A \end{bmatrix}$$

where A is $(-4)^{(n-2)/4}$. The only invariant factor of M_2 is $(A^2 + 1)/5$. So H' is the cyclic group of order $(1 + ((-4)^{(n-2)/4})^2)/5$.

Theorem 3.2.5 Let F be as in Theorem 3.2.1. If $n \geq 10$ and $n \equiv 2(\text{mod}4)$ then F has order $4n(((-4)^{(n-2)/4})^2 + 1)/5$.

Proof: Suppose $n \geq 10$ and $n \equiv 2(\text{mod}4)$. Then by Theorem 3.2.1, F has H as a subgroup of index 2. By Theorem 3.2.2 the derived subgroup of H has index 2n in H. By Lemma 3.2.9 the order of H' is $(((-4)^{(n-2)/4})^2 + 1)/5$. Therefore the order of F is $4n(((-4)^{(n-2)/4})^2 + 1)/5$.

Lemma 3.2.10 Let H' be as in Theorem 3.2.2. If $n \geq 11$ and $n \equiv 3(\text{mod}4)$ then

(i) H' can be generated by x_0 and x_1 and H' has the following presentation:

$$H' = < x_0, x_1 \mid x_1^{(-1+6(-4)^{(n-3)/4})/5} x_0^{(-2+2(-4)^{(n-3)/4})/5} = 1,$$
$$x_1^{(3-8(-4)^{(n-3)/4})/5} x_0^{(1+4(-4)^{(n-3)/4})/5} = 1,$$
$$x_1^{(-2+2(-4)^{(n-3)/4})/5} x_0^{(1-6(-4)^{(n-3)/4})/5} = 1, \ [x_1, x_0] = 1 >$$

(ii) Its order is given by the invariant factors of the relation matrix

$$M_3 = \begin{bmatrix} -1/5 + 6/5A & -2/5 + 2/5A \\ 3/5 - 8/5A & 1/5 + 4/5A \\ -2/5 + 2/5A & 1/5 - 6/5A \end{bmatrix}$$

where A is $(-4)^{(n-3)/4}$.

Proof: (i) Here we are considering the relations for H' given in Theorem 3.2.2 . Using the relation R_{n-2} we can get

$$x_2 = x_1^{-1} \ \dots\dots\dots\dots\dots\dots \text{(i)}.$$

From relation R_1 and using (i) and $[x_1, x_0] = 1$ we get

$$x_3 = x_1 x_0^2 \ \dots\dots\dots\dots\dots\dots \text{(ii)}.$$

Using the relations R_i and formulae given in Lemma 3.2.6, for $i = 1, \dots, n-4$ we can see by induction that

for $t = 1, \dots, (n-3)/4$, $\qquad x_{4t} = x_1^{(1-(-4)^t)/5} x_0^{(2+3(-4)^t)/5} \qquad \dots\dots\dots \text{(At)}.$

for $t = 1, \dots, (n-3)/4$, $\quad x_{4t+1} = x_1^{(1+4(-4)^t)/5} x_0^{(2-2(-4)^t)/5}$ .... (Bt)

for $t = 1, \dots, (n-7)/4$, $\quad x_{4t+2} = x_1^{(1-6(-4)^t)/5} x_0^{(2-2(-4)^t)/5}$ ....(Ct)

for $t = 1, \dots, (n-7)/4$, $\quad x_{4t+3} = x_1^{(1+4(-4)^t)/5} x_0^{(2+8(-4)^t)/5}$ .... (Dt).

So the generators $x_2, x_3, \dots, x_{n-2}$ can be replaced with their new values in terms of x_1 and x_0. As in Lemma 3.2.7 it can be shown that the relations $R_i = 1$, $i = 1, 2, \dots, n-4$ are the trivial relations. It is obvious that for every i, $i = 1,2, \dots n-2$ the relations $R_{n+i} = [x_i, x_{i-1}]$ are the trivial relations. The relation R_{n-2} is trivial which can be shown using (i). The remaining relations are :

$$R_{n+1} = [x_1, x_0],$$
$$R_{n-3} = x_1^{(-1+6(-4)^{(n-3)/4})/5} x_0^{(-2+2(-4)^{(n-3)/4})/5} = 1,$$
$$R_{n-1} = x_1^{(3-8(-4)^{(n-3)/4})/5} x_0^{(1+4(-4)^{(n-3)/4})/5} = 1,$$
$$R_n = x_1^{(-2+2(-4)^{(n-3)/4})/5} x_0^{(1-6(-4)^{(n-3)/4})/5} = 1,$$

So the presentation for H' will be as claimed.

(ii) From **(i)** we can see that H' is abelian. Therefore its order is given by the invariant factors of the relation matrix

$$M_3 = \begin{bmatrix} -1/5 + 6/5A & -2/5 + 2/5A \\ 3/5 - 8/5A & 1/5 + 4/5A \\ -2/5 + 2/5A & 1/5 - 6/5A \end{bmatrix}$$

where A is $(-4)^{(n-3)/4}$. The only invariant factor of M_3 is $(1 - 4A + 8A^2)/5$. So H' is the cyclic group of order $(1 - 4(-4)^{(n-3)/4} + 8((-4)^{(n-3)/4})^2)/5$.

Theorem 3.2.6 Let F be as in Theorem 3.2.1. If $n \geq 11$ and $n \equiv 3(\text{mod}4)$ then F has order $4n(1 - 4(-4)^{(n-3)/4} + 8((-4)^{(n-3)/4})^2)/5$.

Proof: Suppose $n \geq 11$ and $n \equiv 3(\text{mod}4)$. Then by Theorem 3.2.1, F has H as a subgroup of index 2. By Theorem 3.2.2 the derived subgroup of H has index 2n in H. By Lemma 3.2.10 the order of H' is $(1 - 4(-4)^{(n-3)/4} + 8((-4)^{(n-3)/4})^2)/5$. Therefore the order of F is $4n(1 - 4(-4)^{(n-3)/4} + 8((-4)^{(n-3)/4})^2)/5$.

CASE(8):

Theorem 3.2.7 Let $F = G(n; 1, 1, 2, 1, 1, -2) =$

$$\langle A,B \mid A^4 = 1, B^n = 1, ABA^2BAB^{-2} = 1 \rangle.$$

(i) If n is even then F is an infinite group.

(ii) If n is odd then the order of F is 4n.

Proof: (i); Adding the relations $A^2 = 1$ and $B^2 = 1$ to:

$$\langle A,B \mid A^4 = 1, B^n = 1, ABA^2BAB^{-2} = 1 \rangle$$

we get

$$\langle A,B \mid A^2 = 1, B^2 = 1 \rangle$$

which is a presentation for the infinite group $\mathbb{Z}_2 * \mathbb{Z}_2$.

(ii) Let $H = \langle a = B, b = ABA^{-1}, c = A^2BA^{-2}, d = A^3BA^{-3} \rangle$ be a subgroup of F. We define cosets $H = 1$, $1A = 2$, $2A = 3$, $3A = 4$. No collapses occur. Using the Reidemeister - Schreier method we can get the following presentation for H

$$H = \langle a, b, c, d \mid a^n = 1, b^n = 1, c^n = 1, d^n = 1, acd^{-2} = 1, bda^{-2} = 1, cab^{-2} = 1,$$
$$dbc^{-2} = 1, d^{-1} = ab^{-1}a^{-1}, a^{-1} = bc^{-1}b^{-1} \rangle.$$

Considering the relation $d^{-1} = ab^{-1}a^{-1}$ we see that $d = aba^{-1}$.

Eliminating d we get

$$H = \langle a, b, c \mid a^n = 1, b^n = 1, c^n = 1, cab^{-2} = 1,$$
$$baba^{-3} = 1, aba^{-1}bc^{-2} = 1, a^{-1} = bc^{-1}b^{-1} \rangle.$$

From the relation $a^{-1} = bc^{-1}b^{-1}$ we see that $a = bcb^{-1}$. Eliminating a we get

$$H = \langle b, c \mid b^n = 1, c^n = 1, cbcb^{-3} = 1, bcbc^{-3} = 1 \rangle.$$

Using relations 3 and 4 i.e.

$bcbc = b^4$

$bcbc = c^4$

we get $b^4 = c^4$ .... (i).

Now since n is odd therefore $(n,4) = 1$ ...... (ii).

Using (i) and (ii) we get $b = c$.

Therefore H is a cyclic group. Obviously the order of H is n.

Since F has H as a subgroup of index 4 and the order of H is n. Therefore the order of F is 4n.

CASE (7):

Theorem 3.2.8 Let

$F_1 = G(n; 2, 1, 2, 1, 2, -2) = \langle A,B \mid A^4 = 1, B^n = 1, A^2BA^2BA^2B^{-2} = 1 \rangle$

$F_2 = G(n; 2, i, 2, j, 2, k) = \langle A,B \mid A^4 = 1, B^n = 1, A^2B^iA^2B^jA^2B^k = 1 \rangle$.

(i) F_1 is an infinite group for $n > 0$.

(ii) F_2 is an infinite group for $n > 0$, $i+j+k = 0$.

Proof: (ii) Adding the relation $A^2 = 1$ to

$$\langle A,B \mid A^4 = 1, B^n = 1, A^2B^iA^2B^jA^2B^k = 1 \rangle$$

we get

$$\langle A,B \mid A^2 = 1, B^n = 1 \rangle, \text{ since } i+j+k = 0$$

which is a presentation for the infinite group $\mathbb{Z}_2 * \mathbb{Z}_n$.

Proof: (i) It is a special case of **(ii)**.

CASE (1):

Lemma 3.2.11 Let $F = G(n; 1, 1, 1, 1, 1, -2) =$

$$\langle A,B \mid A^4 = 1, B^n = 1, ABABAB^{-2} = 1 \rangle.$$

F has $H = \langle a_1, a_2, a_3, \dots, a_n \mid a_i^4 = 1, a_ia_{i+1}a_{i+2} = 1 \rangle$ as a subgroup of index n, where $a_1 = A$, $a_2 = BAB^{-1}$, $a_3 = B^2AB^{-2}$, ... , $a_n = B^{n-1}AB^{1-n}$.

Proof: This is easily seen using coset enumeration. Let $H = \langle A, BAB^{-1}, B^2AB^{-2}, \dots, B^{n-1}AB^{1-n} \rangle$ where the generators are named $a_1, a_2, a_3, \dots, a_n$. We define cosets $iB = i+1$ for i from 1 to (n-1).
Then the generator a_i gives that $iA = i$ for i from 1 to n. The relation $B^n = 1$ implies that $nB = 1$ and the other relations give the full presentation as claimed .

Lemma 3.2.12 Let H to be as in Lemma 3.2.11. Then

(i) If $n \equiv 0 (\mathrm{mod} 3)$ then H is an infinite group.

(ii) If $n \not\equiv 0 (\mathrm{mod} 3)$ then H is a trivial group.

Proof: Note that

$$a_3 = a_2^{-1} a_1^{-1}$$
$$a_4 = a_3^{-1} a_2^{-1}$$
$$a_4 = a_1$$
$$a_5 = a_4^{-1} a_3^{-1}$$
$$a_5 = a_2$$
$$a_6 = a_5^{-1} a_4^{-1}$$
$$a_6 = a_2^{-1} a_1^{-1}$$
$$a_7 = a_6^{-1} a_5^{-1}$$
$$a_7 = a_1$$
$$a_8 = a_7^{-1} a_6^{-1}$$
$$a_8 = a_2$$
$$a_9 = a_8^{-1} a_7^{-1}$$
$$a_9 = a_2^{-1} a_1^{-1}$$

and it can be easily seen that in general

$a_m = a_2^{-1} a_1^{-1}$, if $m \equiv 0(\text{mod}3)$

$a_m = a_1$, if $m \equiv 1(\text{mod}3)$

$a_m = a_2$, if $m \equiv 2(\text{mod}3)$.

So the generators $a_3, \dots, a_n$ can be replaced by the new values in terms of a_1 and a_2 to obtain:

$\langle a_1, a_2 \mid a_1^4 = 1,\ a_2^4 = 1,\ (a_2^{-1} a_1^{-1})^4 = 1,\ a_2^{-1}a_1 = 1,\ a_1^2 a_2 = 1 \rangle$, if $n \equiv 1(\text{mod}3)$,

$\langle a_1, a_2 \mid a_1^4 = 1,\ a_2^4 = 1,\ (a_2^{-1} a_1^{-1})^4 = 1,\ a_1^2 a_2 = 1,\ a_2^2 a_1 = 1 \rangle$, if $n \equiv 2(\text{mod}3)$,

$\langle a_1, a_2 \mid a_1^4 = 1,\ a_2^4 = 1,\ (a_2^{-1} a_1^{-1})^4 = 1 \rangle$, if $n \equiv 0(\text{mod}3)$.

Case $n \equiv 1(\text{mod}3)$: From relation 4 we can get $a_1 = a_2$ .... (i). Therefore H is a cyclic group. Using (i) in relation 5 we can get $a_1^3 = 1$, but from relation 1, $a_1^4 = 1$ so H is a trivial group.

Case $n \equiv 2(\text{mod}3)$: From relation 4 we get $a_2 = a_1^{-2}$ .... (i). Therefore H is a cyclic group. Using (i) in relation 5 we get $a_1^3 = 1$, but from relation 1, $a_1^4 = 1$ so H is a trivial group.

Case $n \equiv 0(\text{mod}3)$: H is the Von Dyck group(4,4,4) which is an infinite group.

Theorem 3.2.9 Let F be as in Lemma 3.2.11 and H be its subgroup as in Lemma 3.2.11. Then

(i) If $n \equiv 0(\text{mod}3)$ then F is an infinite group.

(ii) If $n \not\equiv 0(\text{mod}3)$ then F is a cyclic group of order n.

Proof: (i) By Lemma 3.2.11, F has H as a subgroup of index n. If $n \equiv 0(\text{mod}3)$ then by Lemma 3.2.12, H is an infinite group. Therefore F is infinite.

(ii) By Lemma 3.2.11, F has H as a subgroup of index n. If $n \not\equiv 0(\text{mod}3)$ then by Lemma 3.2.12 H is a trivial group. Therefore the order of F is n. On the other hand abelianizing the relations of F we see that the factor group has order n. This means F is an abelian group and furthermore F is a cyclic group.

CASE (6):

Lemma 3.2.13 Let $F = G(n; -1, 1, 2, 1, 2, -2) =$

$$\langle A,B \mid A^4 = 1, B^n = 1, A^{-1}BA^2BA^2B^{-2} = 1 \rangle.$$

F has $H = \langle a_1, a_2, a_3, \dots, a_n \mid a_i^4 = 1, a_i = a_{i+1}^2 a_{i+2}^2, i = 1, 2, \dots, n \rangle$ as a subgroup of index n, where $a_1 = A, a_2 = BAB^{-1}, a_3 = B^2AB^{-2}, \dots, a_n = B^{n-1}AB^{1-n}$.

Proof: This can be easily seen using coset enumeration. Let $H = \langle A, BAB^{-1}, B^2AB^{-2}, \dots, B^{n-1}AB^{1-n} \rangle$ where the generators are named $a_1, a_2, a_3, \dots, a_n$. We define cosets $iB = i+1$ for i from 1 to (n-1). Then the generator a_i gives that $iA = i$ for i from 1 to n. The relation $B^n = 1$ implies that $nB = 1$ and other relations give the full presentation as claimed .

Lemma 3.2.14 Let H be as in Lemma 3.2.13. Then for $n \geq 3$, H is a trivial group.

Proof: For n = 3,4,5 and 6 the proof for each case will be given individually, but for $n \geq 7$ the proof will be given in general.

Case n = 3 : If n = 3 then by Lemma 3.2.13 the presentation for H is :

$$\langle a_1, a_2, a_3 \mid a_1^4 = 1,\ a_2^4 = 1,\ a_3^4 = 1,\ a_1 = a_2^2 a_3^2,\ a_2 = a_3^2 a_1^2,\ a_3 = a_1^2 a_2^2 \rangle$$

a_3 is given in terms of a_1 and a_2 as $a_3 = a_1^2 a_2^2$. Eliminating a_3 we get

$$\langle a_1, a_2 \mid a_1^4 = 1,\ a_2^4 = 1,\ (a_1^2 a_2^2)^4 = 1,\ a_1 = a_2^2(a_1^2 a_2^2)^2,\ a_2 = (a_1^2 a_2^2)^2 a_1^2 \rangle$$

Using relation 3 in relation 4 we get $a_1 = a_2^2$ ...... (i).

Using relation 3 in relation 5 we get $a_2 = a_1^2$ .... (ii).

Using (ii) in (i) and using relation 1 we get $a_1 = 1$. .....(iii) . Using (iii) in (ii) we get $a_2 = 1$ so H is a trivial group.

Case n = 4: If n = 4 then by Lemma 3.2.13 the presentation for H is :

$$\langle a_1, a_2, a_3, a_4 \mid a_1^4 = 1,\ a_2^4 = 1,\ a_3^4 = 1,\ a_4^4 = 1,\ a_1 = a_2^2 a_3^2,\ a_2 = a_3^2 a_4^2,\ a_3 = a_4^2 a_1^2,\ a_4 = a_1^2 a_2^2 \rangle$$

a_3 and a_4 can be written in terms of a_1 and a_2 as $a_3 = (a_1^2\, a_2^2)^2 a_1^2$, $a_4 = a_1^2\, a_2^2$.

Eliminating a_3 and a_4 we get

$\langle\, a_1, a_2 \mid a_1^4 = 1,\ a_2^4 = 1,\ ((\, a_1^2\, a_2^2)^2 a_1^2)^4 = 1,\ (a_1^2\, a_2^2)^4,\ a_1 = a_2^2,\ a_2 = (a_1^2 a_2^2)^2 \ \rangle$.

Using relations 5 and 6 we can see that $a_1 = a_2 = 1$. Therefore H is a trivial group.

Case n = 5: If n = 5 then by Lemma 3.2.13 the presentation for H is :

$$\langle\, a_1, a_2, a_3, a_4, a_5 \mid a_1^4 = 1,\ a_2^4 = 1,\ a_3^4 = 1, a_4^4 = 1,\ a_5^4 = 1,\ a_1 = a_2^2\, a_3^2,\ a_2 = a_3^2\, a_4^2,$$
$$a_3 = a_4^2\, a_5^2,\ \ a_4 = a_5^2\, a_1^2,\ \ a_5 = a_1^2\, a_2^2\ \ \rangle.$$

a_4 , a_5 and a_3 can be written in terms of a_1 and a_2 as $a_4 = (a_1^2\, a_2^2)^2 a_1^2$,

$a_5 = a_1^2\, a_2^2$, $a_3 = (a_1^2\, a_2^2)^2$. Eliminating a_3 , a_4 and a_5 we get

$$\langle\, a_1, a_2 \mid a_1^4 = 1,\ a_2^4 = 1, ((a_1^2 a_2^2)^2)^4 = 1,\ ((\, a_1^2\, a_2^2)^2 a_1^2)^4 = 1,\ \ (a_1^2\, a_2^2)^4 = 1,$$
$$a_1 = a_2^2 (a_1^2\, a_2^2)^2,\ \ a_2 = (a_1^2 a_2^2)^4\ \ \rangle .$$

Using relation 5 in relation 7 we see that $a_2 = 1$....... (i). Using (i) and relation 1 in the relation 6 we get $a_1 = 1$. Therefore H is a trivial group.

Case n = 6: If n = 6 then by Lemma 3.2.13 the presentation for H is:

$$\langle\, a_1, a_2, a_3, a_4, a_5, a_6 \mid a_1^4 = 1,\ a_2^4 = 1,\ a_3^4 = 1, a_4^4 = 1\ , a_5^4 = 1,\ \ a_6^4 = 1,$$
$$a_1 = a_2^2\, a_3^2,\ a_2 = a_3^2\, a_4^2,$$
$$a_3 = a_4^2\, a_5^2,\ \ a_4 = a_5^2\, a_6^2,\ \ a_5 = a_6^2\, a_1^2,\ a_6 = a_1^2\, a_2^2\ \ \rangle.$$

a_6 , a_5 , a_4 and a_3 can be written in terms of a_1 and a_2 as $a_6 = a_1^2\, a_2^2$, $a_5 = (a_1^2\, a_2^2)^2 a_1^2$,

$a_4 = (a_1^2\, a_2^2)^2$, $a_3 = (a_1^2\, a_2^2)^4$ Eliminating a_3 , a_4, a_5, a_6 we get

$$\langle\, a_1, a_2 \mid a_1^4 = 1,\ a_2^4 = 1, ((a_1^2 a_2^2)^4)^4 = 1,\ \ ((a_1^2 a_2^2)^2)^4 = 1\ ((\, a_1^2\, a_2^2)^2 a_1^2)^4 = 1,\ \ (a_1^2\, a_2^2)^4 = 1,$$
$$a_1 = a_2^2 (a_1^2\, a_2^2)^4,\ \ a_2 = (a_1^2 a_2^2)^4\ \ \rangle .$$

Using relation 6 in relation 8 we see that $a_2 = 1$....... (i). Using (i) and relation 1 in relation 7 we get $a_1 = 1$. Therefore H is a trivial group.

Case n = 7: If $n \geq 7$ then by Lemma 3.2.13 the presentation for H is :

$H = \langle\, a_1, a_2, a_3, \ldots , a_n \mid a_i^4 = 1, a_i = a_{i+1}^2\, a_{i+2}^2\ ,\ i = 1, 2, \ldots , n\ \rangle$.

a_3 , a_4 , ... , a_n can be written in terms of a_1 and a_2 as

$a_n = a_1^2\, a_2^2$,

$a_{n-1} = (a_1^2 a_2^2)^2 a_1^2,$

$a_{n-2} = (a_1^2 a_2^2)^{g_1},$

"

"

$a_3 = (a_1^2 a_2^2)^{g_{n-4}}$

where $g_1 = 2,\ g_2 = 4,\ g_i = 2(g_{i-1} + g_{i-2})$

Eliminating $a_3, a_4, \ldots, a_n$ we get the following presentation for H

$$\langle a_1, a_2 \mid a_1^4 = 1,\ a_2^4 = 1,\ ((a_1^2 a_2^2)^{g_{n-4}})^4 = 1,\ ((a_1^2 a_2^2)^{g_{n-5}})^4 = 1,$$
$$((a_1^2 a_2^2)^{g_{n-6}})^4 = 1, \ldots, ((a_1^2 a_2^2)^{g_1})^4 = 1, ((a_1^2 a_2^2)^2 a_1^2)^4 = 1,\ (a_1^2 a_2^2)^4 = 1,$$
$$a_1 = a_2^2 (a_1^2 a_2^2)^{2g_{n-4}},\ a_2 = (a_1^2 a_2^2)^{2g_{n-4}} (a_1^2 a_2^2)^{2g_{n-5}} \rangle$$

Using relation n in relation n+2 we see that $a_2 = 1$....... (i). Using (i) and relation 1 in the relation n+1 we get $a_1 = 1$. Therefore H is a trivial group. Here we can notice that $g_1 = 2,\ g_2 = 4$ and for $i \geq 3$, g_i is divisible by 4, by the definition of g_i.

Theorem 3.2.10 Let F be as in Lemma 3.2.13 and H be its subgroup as in Lemma 3.2.13. Then the order of F is n.

Proof: By Lemma 3.2.13 F has H as a subgroup of index n. By Lemma 3.2.14 H is a trivial group. Therefore the order of F is n. On the other hand abelianizing the relations of F we see that the factor group has order n. This means F is an abelian group. Furthermore F is a cyclic group.

CASE (10):

Theorem 3.2.12 Let

$F = G(n;\ 1, 1, 2, 1, -1, -2) = \langle A, B \mid A^4 = 1, B^n = 1, ABA^2BA^{-1}B^{-2} = 1 \rangle$

If n is even then F is an infinite group.

Proof: Adding the relations $B^2 = 1$, $A^2 = 1$ to

$$\langle A,B \mid A^4 = 1, B^n = 1, ABA^2BA^{-1}B^{-2} = 1 \rangle$$

we get

$$\langle A,B \mid A^2 = 1, B^2 = 1 \rangle$$

which is a presentation for the infinite group $\mathbb{Z}_2 * \mathbb{Z}_2$.

For small values of n we see using computer proofs that the following conjectures are true. Therefore we conjecture they are true for all n.

Conjecture; if n is odd then F, given in Theorem 3.2.12, is an abelian group of order 2n.

Conjecture for case (5): Let

$F = G(n; 1, 1, -1, 1, 2, -2) = \langle A,B \mid A^4 = 1, B^n = 1, ABA^{-1}BA^2B^{-2} = 1 \rangle$.

If n is odd then F has order 2n.

If n is even then F has order $4n.(2^{n/2} - 1)^2$.

CHAPTER 4

DIFFERENT EFFICIENT PRESENTATIONS FOR CERTAIN OF THE GROUPS PSL(2,p)

4.0 INTRODUCTION:

This chapter will consist of two sections. In section one different efficient presentations will be given for the groups PSL(2,p), where p is an odd prime number and $p \in \{ 5,7,11,13,17,19,23,29,31,37,41,43,53,59,79,83,89,109,139,229 \}$.

In section two different efficient presentations will be given for the groups PSL(2,p), where p is a prime power and $p \in \{ 9,25,27,49,169 \}$.

Our actual aim is to investigate the structure of the groups in the classes given in these two sections. The results for each class will be given under a theorem.

In the proof of the theorems throughout this chapter the following CAYLEY program 1 has been used. The program was written to investigate when two groups are isomorphic. A brief description of the idea behind the program is outlined below.

Suppose g and f represent two different groups where g is our group whose structure is not known and f is the group whose structure is known and is generated by two elements x and y of order n and m respectively subject to two further defining relations. Now we are taking $h=<h_1, h_2, h_3>$ as a subgroup of g, where h_1,h_2,h_3 are generators for h in terms of generators of g. Using standard function cosact homomorphism we construct the permutation representation of g on the cosets of h.

Let us use i to denote the image of g under the cosact homomorphism function. Running through i we collect elements of order n up to the certain number N in the set s and elements of order m up to the certain number M in the set t. Suppose the cardinality of s is greater than the cardinality of t. Now we are running at the same time through the sets s and t to check if any pair $z \in s$, $q \in t$ satisfy the relations of f and if $| \langle z, q \rangle | = | \langle i \rangle |$. If so this means the group g and the group f are isomorphic. If we can't find any z,q pair to satisfy the conditions for isomorphism in the first run then we can increase the number of collected elements in the sets s and t. Then again we can check if the conditions for isomorphism have been met. If not we again continue to increase the numbers of collected elements in sets s and t and check again. In the end we always finish by finding such a pair and we have made the program to print this pair.

We are not going to give these pairs in order to save space. However, they can be printed using the program.

The following standard functions are used in the program.

g.relations:	defining relations for the group g which are satisfied by the generators of g
f.relations:	defining relations for the group f which are satisfied by the generators of f
index(g,h):	given a subgroup h of the finitely presented group g, this function returns the index of h in g as an integer
order(i):	order of the group i
satisfy (Q,W):	given a sequence Q and n elements belonging to a group g and a set of words W on the generators of the n-generator group h, returns the Boolean value true if every member of W is the identity of h under the substitution $h.i \rightarrow Q[i]$ ($i = 1,2,...,n$)

cosact

homomorphism(g,h): this function constructs the permutation of g given by the action of g on the cosets of h.

The notation which has been used in the program is as follows:

g	will denote the related group $F_i/(\langle b^r \rangle = H_i)$
f	will denote the related PSL(2,p)
h	will denote the related K_i
n	will denote the order of 1st generator of the related PSL(2,p)
m	will denote the order of second generator of the related PSL(2,p)
s	a set of elements of order n
t	a set of elements of order m
R_{g3}(a,b)	will denote the 3rd relation of the group $F_i/(\langle b^r \rangle = H_i)$
R_{g4}(a,b)	will denote the 4th relation of the group $F_i/(\langle b^r \rangle = H_i)$
R_{f3}(x,y)	will denote the 3rd relation of the related PSL(2,p)
R_{f4}(x,y)	will denote the 4th relation of the related PSL(2,p) (if the group has a 4th relation)
card(s)	cardinality of the set s
card(t)	cardinality of the set t

CAYLEY program 1:

```
>set workspace=2000000;
>g:free(a,b);
>g.relations: a^c,b^d,Rg3(a,b),Rg4(a,b);
>f:free(x,y);
>f.relations:x^n,y^m,Rf3(x,y),Rf4(x,y);
```

```
>h=<h1,h2,h3>;
>p,i,k=cosact homomorphism(g,h);
>PRINT ' index of h in g is ' , index(g,h);
>PRINT ' order of i is ' , order(i);
>s=null;
>t=null;
>j=0;
>  FOR EACH v IN  i  DO
>o=order(v);
>     IF o EQ n
>        THEN
>           s=s join[v];
>     END;
>     IF  o EQ m
>        THEN
>           t=t join[v];
>     END;
>  j=j+1;
>     IF  (j mod 10)  EQ  0
>        THEN
>           PRINT  j ,  card(s) ,  card(t);
>     END;
>     IF  (card(s))  GT  N and (card(t))  GT M
>        THEN
>           IF  ((card(t))  GT  (card(s)))  OR  ((card(s))  EQ  (card(t)))
>              THEN
```

```
>           FOR EACH  z  IN  t  DO
>             FOR EACH  q  IN  s  DO
>               IF satisfy(SEQ(q,z),relations(f)) and order(<q,z>) eq order(i)
>                 THEN
>                   PRINT ' isomorphism  true ' ;
>                   PRINT  ' element z ' , z;
>                   PRINT ' order of z is ' , order(z);
>                   PRINT ' element q ' , q;
>                   PRINT ' order of q is ' , order(q);
>                   PRINT ' cardinal of s is ' , (card(s));
>                   PRINT ' cardinal of t is ' , (card(t));
>                   PRINT ' order of q^n is ' , order(q^n);
>                   PRINT ' order of z^m is ' , order(z^m);
>                   PRINT ' order of (Rf3(q,z)) is ' , order((Rf3(q,z)));
>                   PRINT ' order of (Rf4(q,z)) is ' , order((Rf4(q,z)));
>                   PRINT ' order of <q,z> is ' , order(<q,z>);
>                 QUIT;
>               END;
>             END;
>       END;
>   END;
>   IF  (card(s))  GT  (card(t))
>     THEN
>       FOR EACH  z  IN  s  DO
>         FOR EACH  q  IN  t  DO
>           IF satisfy(SEQ(z,q),relations(f)) and order(<z,q>) eq order(i)
```

```
>                THEN
>                   PRINT ' isomorphism true ' ;
>                   PRINT ' element z is ' , z;
>                   PRINT ' order of z is ' , order(z);
>                   PRINT ' element q is ' , q;
>                   PRINT ' order of q is ' , order(q);
>                   PRINT ' cardinal of s is ' , (card(s));
>                   PRINT ' cardinal of t is ' , (card(t));
>                   PRINT ' order of z^n is ' , order(z^n);
>                   PRINT ' order of q^m is ' , order(q^m);
>                   PRINT ' order of (Rf3(z,q)) is ' , order((Rf3(z,q)));
>                   PRINT ' order of (Rf4(z,q)) is ' , order((Rf4(z,q)));
>                   PRINT ' order of <z,q> is ' , order(<z,q>);
>              QUIT;
>           END;
>        END;
>    END;
>  END;
>   PRINT ' not isomorphic ' ;
>   PRINT ' cardinal of s is ' , card(s);
>   PRINT ' cardinal of t is '  , card(t);
>  QUIT;
>  END;
>END;
```

Presentations for related PSL(2,p) groups, in Theorems 4.1.1 – 4.1.20, have been taken from the well known presentation due to C.M.Campell and E.F.Robertson

[6]. The presentation for the related PSL(2,5) group in Theorem 4.2.1 has been taken from A. Jamali [21]. Presentations for related PSL(2,p) groups in Theorems 4.2.2 – 4.2.4 have been taken from C.M.Campbell, E.F.Robertson al [9]. The presentation for the related PSL(2,169) group in Theorem 4.2.5 has been taken from C.M.Campbell, E.F.Robertson and P.D.Williams [12].

Throughout this chapter the following CAYLEY program 2 will be used. The program was written to obtain a permutation representation for each of the F_i groups.

We are going to briefly describe the idea behind the program.

Suppose g is the group whose structure is known and it is generated by two elements a and b of orders c and d respectively and g has a^c, $R_2(a,b)$, $R_3(a,b)$ as defining relations. Now we are taking $h=\langle h_1,h_2,h_3\rangle$ as a subgroup of g, where h_1,h_2,h_3 are generators for h in terms of generators of g. Using standard function cosact homomorphism we construct the permutation representation of g on the cosets of h. Let us i to denote the image of g under the cosact homomorphism function. Running through i we collect elements of order c up to a certain number N, in the set s and elements of order d up to a certain number M, in the set t. Suppose cardinality of s is greater than cardinality of t. Now we are running at the same time through the sets s and t to check if any pair $z \in s$, $q \in t$ satisfy the relations of g and if $| \langle z,q \rangle | = | \langle i \rangle |$. If so this means we have obtained a permutation pair which generates g and satisfies the relations of g. If we can not find such a pair we can increase the numbers of collected elements in sets s and t and check again. In the end we always finish by finding such a pair.

The following standard functions are used in the program.

g.relations:	defining relations for the group g which are satisfied by the generators of g
index(g,h):	given a subgroup h of the finitely presented group g, this function returns the index of h in g as an integer
order(i):	order of the group i
satisfy (Q,W):	given a sequence Q and n elements belonging to a group g and a set of words W on the generators of the n-generator group h, returns the Boolean value true if every member of W is the identity of h under the substitution h.i $\rightarrow$ Q[i] (i = 1,2,...,n)
cosact	
homomorphism(g,h):	this function constructs the permutation of g given by the action of g on the coset of h.

The notation which has been used in the program is as follows:

g	will denote the related group F_i
h	will denote the maximal subgroup of the related group F_i
c	will denote the order of the 1st generator of the related group F_i
d	will denote the order of the second generator of the related group F_i
s	a set of elements of order c
t	a set of elements of order d
a^c	will denote the first relation of the related group F_i
$R_{g2}(a,b)$	will denote the second relation of the related group F_i
$R_{g3}(a,b)$	will denote the 3rd relation of the related group F_i
card(s)	cardinality of the set s
card(t)	cardinality of the set t

CAYLEY program 2:

```
>set workspace=2000000;
>g:free(a,b);
>g.relations: a^c,R2(a,b),R3(a,b);
>h=<h1,h2,h3>;
>p,i,k=cosact homomorphism(g,h);
>PRINT ' index of h in g is ' , index(g,h);
>PRINT ' order of i is ' , order(i);
>s=null;
>t=null;
>j=0;
>   FOR EACH v IN  i  DO
>o=order(v);
>       IF o EQ c
>          THEN
>              s=s join[v];
>       END;
>       IF  o EQ d
>          THEN
>              t=t join[v];
>       END;
>   j=j+1;
>       IF  (j mod 10)  EQ  0
>          THEN
>              PRINT  j ,  card(s) ,  card(t);
>       END;
```

```
>    IF (card(s)) GT N and (card(t)) GT M
>      THEN
>        IF ((card(t)) GT (card(s))) OR ((card(s)) EQ (card(t)))
>          THEN
>            FOR EACH z IN t DO
>              FOR EACH q IN s DO
>                IF satisfy(SEQ(q,z),relations(g)) and order(<q,z>) eq order(i)
>                  THEN
>                    PRINT ' (q,z) satisfy g ';
>                    PRINT  ' element z ' , z;
>                    PRINT ' order of z is ' , order(z);
>                    PRINT ' element q ' , q;
>                    PRINT ' order of q is ' , order(q);
>                    PRINT ' cardinal of s is ' , (card(s));
>                    PRINT ' cardinal of t is ' , (card(t));
>                    PRINT ' order of q^c is ' , order(q^c);
>                     PRINT ' order of (R2(q,z)) is ' , order((R2(q,z)));
>                     PRINT ' order of (R3(q,z)) is ' , order((R3(q,z)));
>                     PRINT ' order of <q,z> is ' , order(<q,z>);
>                  QUIT;
>                END;
>              END;
>        END;
>    END;
>    IF (card(s)) GT (card(t))
>      THEN
```

```
>           FOR EACH  z  IN  s  DO
>             FOR EACH  q  IN  t  DO
>               IF satisfy(SEQ(z,q),relations(g)) and order(<z,q>) eq order(i)
>                 THEN
>                   PRINT ' (z,q) satisfy g ' ;
>                   PRINT ' element z is ' , z;
>                   PRINT ' order of z is ' , order(z);
>                   PRINT ' element q is ' , q;
>                   PRINT ' order of q is ' , order(q);
>                   PRINT ' cardinal of s is ' , (card(s));
>                   PRINT ' cardinal of t is ' , (card(t));
>                   PRINT ' order of z^c is ' , order(z^c);
>                   PRINT ' order of (R2(z,q)) is ' , order((R2(z,q)));
>                   PRINT ' order of (R3(z,q)) is ' , order((R3(z,q)));
>                   PRINT ' order of <z,q> is ' , order(<z,q>);
>               QUIT;
>             END;
>           END;
>      END;
>   END;
>    PRINT ' g is not satisfied by any pair ' ;
>    PRINT ' cardinal of s is ' , card(s);
>    PRINT ' cardinal of t is '  , card(t);
>   QUIT;
>  END;
>END;
```

4.1 DIFFERENT EFFICIENT PRESENTATIONS FOR THE GROUPS PSL(2,p) where p belongs to the set { 5,7,11,13,17,19,23,29,31,37,41,43,53,59,79,83,89,109,139,229 } :

Considerable attention has been given to the problem of finding economical presentations for the groups PSL(2,p). The first abstract presentation to appear in the literature was a 2-generator 3-relator presentation for PSL(2,5) and was given by Hamilton in 1856. Therefore PSL(2,5) has long been known to be efficient. For the groups PSL(2,p), p prime p>5, Zassenhaus in [43] gave a 2-generator 3-relation presentation for the first time. A few years later Sunday in [36] gave a neater presentation for them. Later in [7] C.M.Campbell and E.F.Robertson gave the most symmetric presentations so far obtained. Their presentation was based on a presentation due to Beetham [3] and a reduction of this presentation in [34].

In this section we study groups of the following types

(i) $\langle a,b \mid a^k = (ab)^2 = b^{m+s}a^{-n}b^ma^{-n} = 1 \rangle$

(ii) $\langle a,b \mid a^k = (ab)^2 = b^{m+s}a^{-n}(b^ma^{-n})^2 = 1 \rangle$

(iii) $\langle a,b \mid a^k = (a^2b^{-1})^2 = b^{m+s}a^{-n}(b^ma^{-n})^3 = 1 \rangle$

(iv) $\langle a,b \mid a^k = (a^2b^{-2})^2 = b^{m+s}a^nb^ma^nb^ma^n = 1 \rangle$

In this section we are going to show that for certain values of k,s,m,n these groups are efficient and isomorphic to the groups PSL(2,p).

Theorem 4.1.1 If ($m \equiv 2(\mathrm{mod}5)$ and $n = 2$) or ($m \equiv 3(\mathrm{mod}5)$ and $n = 3$) then

$$F_1 = \langle a,b \mid a^5 = (ab)^2 = b^{m+5}a^{-n}b^m a^{-n} = 1 \rangle \cong PSL(2,5)$$

and this is an efficient presentation for F_1.

Proof: Case $m \equiv 2(\mathrm{mod}5)$ and $n = 2$;

Let $m \equiv 2(\mathrm{mod}5)$ and $n = 2$ then $F_1 = F'_1$ and consequently $b^5 \in F_1 = F'_1$. From relation 3 and using relation 1 it can be easily seen that b^5 commutes with a^2. The order of a is odd. Hence b^5 commutes with a, and so $b^5 \in Z(F_1)$. Since $b^5 \in F'_1$ we see that $b^5 \in Z(F_1) \cap F'_1$.

Consider the homomorphic image of F_1 by $H_i = \langle b^5 \rangle$ i.e.

$$F_1/\langle b^5 \rangle = \langle a,b \mid a^5 = (ab)^2 = (b^2a^{-2})^2 = b^5 = 1 \rangle$$

Now take $K_i = \langle a^{-1}ba^{-1}, a^{-2}b \rangle < F_1/\langle b^5 \rangle$.

Using the CAYLEY program 1 it can be shown that the permutation representation of $F_1/\langle b^5 \rangle$ on the cosets of K_i is isomorphic to PSL(2,5).

Using TC on $L_1 = \langle a \rangle < F_1/\langle b^5 \rangle$ it can be verified that the order of $F_1/\langle b^5 \rangle$ is equal to the order of the group PSL(2,5), hence $F_1/\langle b^5 \rangle \cong PSL(2,5)$.

Since $b^5 \in Z(F_1) \cap F'_1$ we can deduce that $\langle b^5 \rangle \leq M(PSL(2,5)) = C_2$. This means $| \langle b^5 \rangle | = 1$ or 2. i.e. $F_1 \cong PSL(2,5)$ or F_1 is isomorphic to its covering group, SL(2,5).

Assume $(b^5)^2 = 1$ but $b^5 \neq 1$. i.e. $F_1 \cong SL(2,5)$.

By observation it can be seen that F_1 is generated by a and ab, the latter element having order 2. On the other hand SL(2,5) has only one element of order 2, so this element has to be ab. However SL(2,5) is not generated by the element of order two and one other element. Therefore there is only one possibility which is $F_1 \cong PSL(2,5)$.

F_1 has two generators, three relations and the Schur multiplier of F_1 is C_2. Therefore this presentation for F_1 is efficient.

Case $\mathbf{m \equiv 3 (mod 5)}$ and $\mathbf{n = 3}$: If $m \equiv 3(\mathrm{mod}5)$ and $n = 3$ then

$$F_1 \cong \langle\, a,b \mid a^5 = (ab)^2 = b^{p+5}a^{-q}b^{p}a^{-q} = 1 \,\rangle$$

here $p \equiv 2(\mathrm{mod}5)$ and $q = 2$.

Proof Let $m \equiv 3(\mathrm{mod}5)$ and $n = 3$ then

$$F_1 = \langle\, a,b \mid a^5 = (ab)^2 = b^{m+5}a^{-3}b^{m}a^{-3} = 1 \,\rangle$$

Using the map $b \mapsto b^{-1}$, $a \mapsto a^{-1}$ we can get

$$F_1 = \langle\, a,b \mid a^5 = (ab)^2 = b^{-m-5}a^{3}b^{-m}a^{3} = 1 \,\rangle$$

Now let $-m = p + 5$.

Since $m \equiv 3(\mathrm{mod}5) \Rightarrow p \equiv 2(\mathrm{mod}5)$. Replacing $-m = p + 5$ in F_1 and using relation 1 yields the result as claimed.

In the following nineteen theorems full proofs are essentially the same as the proof of Theorem 4.1.1 with slight modifications. In every case instead of full proofs only the modifications will be given which have to be made in the proof of Theorem 4.1.1 and in the CAYLEY program 1, in order to obtain the full proof. These modifications are:

1. The conditions which make F_i perfect.
2. The proof that b^r commutes with a.
3. The element b^r which is in $Z(F_i) \cap F_i'$.
4. The subgroup H_i which is going to be used to construct the homomorphic image, $F_i/\langle b^r \rangle$.
5. The subgroup K_i which is going to be used to construct the permutation representation of $F_i/\langle b^r \rangle$ on the cosets of K_i.

6. The subgroup L_i which is going to be used to enumerate $F_i/\langle b^r \rangle$.

7. The pair (x,y) which generates the related SL(2,p).

8. The conditions for card(s) and card(t) in the CAYLEY program 1.

Additionally it can be pointed out that actually K_i is a maximal subgroup of F_i with minimal index. Therefore with the CAYLEY program 2, a permutation representation has been obtained for each of the F_i groups. In every case we give these permutation generating pairs.

For Theorem 4.1.1 the permutation generating pair of degree 5 is:

$\alpha = (1,3,4,5,2)$,

$\beta = (1,2,3,4,5)$

and in the CAYLEY program 2 card(s) and card(t) must be greater than 4.

Also in the CAYLEY program 1 card(s) and card(t) must be greater than 1.

Theorem 4.1.2 If $m \equiv 1(\mathrm{mod}3)$, $m \neq 21t + 19$ and $n = 1$ then

$$F_2 = \langle\, a,b \mid a^7 = (a^2b^{-1})^2 = b^{m+3}a^{-n}(b^m a^{-n})^3 = 1 \,\rangle \cong PSL(2,7)$$

and this is an efficient presentation for F_2.

The permutation generating pair of degree 7 is:

$\alpha = (1,2,4,5,6,7,3)$

$\beta = (2,6,3)(4,5,7)$.

In the CAYLEY program 2 card(s) and card(t) must be greater than 1.

Modifications which have to be made in Theorem 4.1.1 are as follows:

1. If $m \equiv 1(\mathrm{mod}3)$, $m \neq 21t + 19$ and $n = 1$ then $F_2 = F_2'$.
2. From relation 3 and using relation 1 it can be easily seen that b^3 commutes with a.

3. If $m \equiv 1(\mathrm{mod}3)$, $m \neq 21t + 19$ and $n = 1$ then $b^3 \in Z(F_2) \cap F'_2$.

4. $H_2 = \langle b^3 \rangle$.

5. $K_2 = \langle a^{-1}ba^{-1}, ba^{-1}b^{-1}a^{-4}b \rangle$.

6. $L_2 = \langle a \rangle$.

7. Generating pair for SL(2,7) is (a, a^2b^{-1}).

8. In the CAYLEY program 1 card(s) and card(t) must be greater than 1.

Theorem 4.1.3 If $m = 10t + 2$, $t \in \mathbb{Z}$ and $n = 2$ then

$$F_3 = \langle a,b \mid a^5 = (ab)^2 = b^{m+5}a^{-n}(b^m a^{-n})^2 = 1 \rangle \cong PSL(2,11)$$

and this is an efficient presentation for F_3.

The permutation generating pair of degree 11 is:

$\alpha = (2,3,5,6,4)(7,11,8,9,10)$,

$\beta = (1,4,10,3,2)(5,9,8,7,6)$.

In the CAYLEY program 2 card(s) and card(t) must be greater than 4.

Modifications which have to be made in Theorem 4.1.1 are as follows:

1. If $m = 10t + 2$, $t \in \mathbb{Z}$ and $n = 2$ then $F_3 = F'_3$.

2. From relation 3 and using relation 1 it can be easily seen that b^5 commutes with a^2. The order of a is odd, so b^5 commutes with a.

3. If $m = 10t + 2$, $t \in \mathbb{Z}$ and $n = 2$ then $b^5 \in Z(F_3) \cap F'_3$.

4. $H_3 = \langle b^5 \rangle$.

5. $K_3 = \langle a^{-1}, b^{-1}ab^2 \rangle$.

6. $L_3 = \langle a \rangle$.

7. Generating pair for SL(2,11) is (a,ab).

8. In the CAYLEY program 1 card(s) and card(t) must be greater than 4.

Theorem 4.1.4 If $m \equiv 2(\mathrm{mod}7)$, $m \neq 91t+9$ and $n = 7$ then

$$F_4 = \langle\, a,b \mid a^{13} = (ab)^2 = b^{m+7}a^{-n}b^{m}a^{-n} = 1 \,\rangle \cong PSL(2,13)$$

and this is an efficient presentation for F_4.

The permutation generating pair of degree 14 is:

$\alpha = (2,12,13,14,3,4,5,6,7,8,9,10,11)$,

$\beta = (1,2,3,14,10,9,12)(4,11,13,8,5,6,7)$.

In the CAYLEY program 2 card(s) and card(t) must be greater than 1.

Modifications which have to be made in Theorem 4.1.1 are as follows:

1. If $m \equiv 2(\mathrm{mod}7)$, $m \neq 91t+9$ and $n = 7$ then $F_4 = F_4'$.
2. From relation 3 and using relation 1 it can be easily seen that b^7 commutes with a.
3. If $m \equiv 2(\mathrm{mod}7)$, $m \neq 91t+9$ and $n = 7$ then $b^7 \in Z(F_4) \cap F_4'$.
4. $H_4 = \langle b^7 \rangle$.
5. $K_4 = \langle a, ba^{-9}b^{-2} \rangle$.
6. $L_4 = \langle a \rangle$.
7. Generating pair for SL(2,11) is (a,ab).
8. In the CAYLEY program 1 card(s) and card(t) must be greater than 1.

Theorem 4.1.5 If ($m \equiv 1(\mathrm{mod}3)$ and $n = 1$) or ($m \equiv 2(\mathrm{mod}3)$ and $n = 8$) then

$$F_5 = \langle\, a,b \mid a^9 = (a^2b^{-1})^2 = b^{m+3}a^{-n}(b^{m}a^{-n})^3 = 1 \,\rangle \cong PSL(2,17)$$

and this is an efficient presentation for F_5.

Case $m \equiv 1(\mathrm{mod}3)$ and $n = 1$:

The permutation generating pair of degree 18 is:

$\alpha = (1,5,17,16,15,14,2,12,13)(3,4,6,7,8,9,11,10,18)$,

$\beta = (1,3,2)(4,8,5)(6,7,15)(9,10,17)(11,12,13)(14,18,16)$.

In the CAYLEY program 2 card(s) and card(t) must be greater than 2.

Modifications which have to be made in Theorem 4.1.1 are as follows:

1. If $m \equiv 1 \pmod 3$ and $n = 1$ then $F_5 = F'_5$.
2. From relation 3 and using relation 1 it can be easily seen that b^3 commutes with a^2. The order of a is odd, so b^3 commutes with a.
3. If $m \equiv 1 \pmod 3$ and $n = 1$ then $b^3 \in Z(F_5) \cap F'_5$.
4. $H_5 = \langle b^3 \rangle$.
5. $K_5 = \langle aba^{-1}b^2, a^{-2}b^{-1}ab^{-1}a^2b^{-1}a^{-1}b^{-1}a^{-1} \rangle$.
6. $L_5 = \langle b \rangle$.
7. Generating pair for SL(2,17) is (a, a^2b^{-1}).
8. In the CAYLEY program 1 card(s) and card(t) must be greater than 1.

Case $m \equiv 2 \pmod 3$ and $n = 8$: If $m \equiv 2 \pmod 3$ and $n = 8$ then

$$F_5 = \langle a,b \mid a^9 = (a^2b^{-1})^2 = b^{p+3}a^{-q}(b^pa^{-q})^3 = 1 \rangle$$

where $p \equiv 1 \pmod 3$ and $q = 1$.

Proof Let $m \equiv 2 \pmod 3$ and $n = 8$ then

$$F_5 = \langle a,b \mid a^9 = (a^2b^{-1})^2 = b^{m+3}a^{-8}(b^ma^{-8})^3 = 1 \rangle$$

Using the map $b \mapsto b^{-1}$, $a \mapsto a^{-1}$ we can get

$$F_5 = \langle a,b \mid a^9 = (a^2b^{-1})^2 = b^{-m-3}a^8(b^{-m}a^8)^3 = 1 \rangle.$$

Now let $-m = p + 3$.

Since $m \equiv 2 \pmod 3 \Rightarrow p \equiv 1 \pmod 3$. Replacing $-m = p + 3$ in F_5 and using relation 1 yields the result as claimed.

Theorem 4.1.6 If $m \equiv 7(\mathrm{mod}9)$, $m \neq 45t + 16$ and $n = 2$ then

$$F_6 = \langle a,b \mid a^5 = (ab)^2 = b^{m+9}a^{-n}b^m a^{-n} = 1 \rangle \cong PSL(2,19)$$

and this is an efficient presentation for F_6.

The permutation generating pair of degree 20 is:

$\alpha = (1,3,4,12,2)(5,13,19,10,11)(6,17,20,16,14)(7,8,15,9,18)$,

$\beta = (1,15,14,20,7,10,11,18,4)(2,5,19,6,8,17,13,12,9)$.

In the CAYLEY program 2 card(s) and card(t) must be greater than 1.

Modifications which have to be made in Theorem 4.1.1 are as follows:

1. If $m \equiv 7(\mathrm{mod}9)$, $m \neq 45t + 16$ and $n = 2$ then $F_6 = F'_6$.
2. From relation 3 and using relation 1 it can be easily seen that b^9 commutes with a^2. The order of a is odd, so b^9 commutes with a.
3. If $m \equiv 7(\mathrm{mod}9)$, $m \neq 45t + 16$ and $n = 2$ then $b^9 \in Z(F_6) \cap F'_6$.
4. $H_6 = \langle b^9 \rangle$.
5. $K_6 = \langle b, a^{-3}b^{-1}ab^{-4}ab^{-3}ab^{-1}a^{-1} \rangle$.
6. $L_6 = \langle a \rangle$.
7. Generating pair for SL(2,19) is (a,ab).
8. In the CAYLEY program 1 card(s) and card(t) must be greater than 1.

Theorem 4.1.7 If $m \equiv 2(\mathrm{mod}3)$, $m \neq 33t + 20$ and $n = 3$ then

$$F_7 = \langle a,b \mid a^{11} = (a^2b^{-1})^2 = b^{m+3}a^{-n}(b^m a^{-n})^3 = 1 \rangle \cong PSL(2,23)$$

and this is an efficient presentation for F_7.

The permutation generating pair of degree 24 is:

$\alpha = (2,18,17,16,19,9,8,7,5,4,3)(10,24,15,14,13,12,22,23,20,21,11)$,

$\beta = (1,2,4)(3,5,6)(7,17,20)(8,18,12)(9,11,10)(13,21,24)(14,16,15)(19,23,22)$.

In the CAYLEY program 2 card(s) and card(t) must be greater than 1.

Modifications which have to be made in Theorem 4.1.1 are as follows:

1. If $m \equiv 2(\mathrm{mod}\,3)$, $m \neq 33t+20$ and $n = 3$ then $F_7 = F'_7$.
2. From relation 3 and using relation 1 it can be easily seen that b^3 commutes with a^3. The order of a is 11, so b^3 commutes with a.
3. If $m \equiv 2(\mathrm{mod}\,3)$, $m \neq 33t+20$ and $n = 3$ then $b^3 \in Z(F_7) \cap F'_7$.
4. $H_7 = \langle b^3 \rangle$.
5. $K_7 = \langle a, ba^3ba^3ba^{-1}baba^3bab^{-1}a^3b^{-1}a^{-1} \rangle$.
6. $L_7 = \langle a \rangle$.
7. Generating pair for SL(2,23) is (a, a^2b^{-1}).
8. In the CAYLEY program 1 card(s) and card(t) must be greater than 1.

Theorem 4.1.8 If $m \equiv 8(\mathrm{mod}\,15)$, $m \neq 105t+8$ and $n = 2$ then

$$F_8 = \langle a,b \mid a^7 = (ab)^2 = b^{m+15}a^{-n}b^{m}a^{-n} = 1 \rangle \cong PSL(2,29)$$

and this is an efficient presentation for F_8.

The permutation generating pair of degree 30 is:

$\alpha = (2,3,4,5,24,11,18)(6,25,22,30,29,23,7)(8,27,28,21,26,10,9)$
$(12,17,20,13,14,15,16)$,

$\beta = (1,18,26,25,4,24,23,14,16,22,21,20,19,17,2)(3,12,13,28,27,7,5,6,8,30,15,29,$
$9,10,11)$.

In the CAYLEY program 2 card(s) and card(t) must be greater than 1.

Modifications which have to be made in Theorem 4.1.1 are as follows:

1. If $m \equiv 8(\mathrm{mod}\,15)$, $m \neq 105t+8$ and $n = 2$ then $F_8 = F'_8$.
2. From relation 3 and using relation 1 it can be easily seen that b^{15} commutes with a^2. The order of a is odd, so b^{15} commutes with a.

3. If $m \equiv 8 (\mathrm{mod} 15)$, $m \neq 105t + 8$ and $n = 2$ then $b^{15} \in Z(F_8) \cap F'_8$.

4. $H_8 = \langle b^{15} \rangle$.

5. $K_8 = \langle a, b^{-2}a^{-5}b^{-5}ab^{-3}ab^{-1}a^{-3}ba^{-1} \rangle$.

6. $L_8 = \langle a \rangle$.

7. Generating pair for SL(2,29) is (a,ab).

8. In the CAYLEY program1 card(s) and card(t) must be greater than 1.

Theorem 4.1.9 If $m \equiv 1 (\mathrm{mod} 8)$, $m \neq 40t + 33$ and $n = 1$ then

$$F_9 = \langle a, b \mid a^5 = (a^2b^{-2})^2 = b^{m+8}a^n b^m a^n b^m a^n = 1 \rangle \cong PSL(2,31)$$

and this is an efficient presentation for F_9.

The permutation generating pair of degree 32 is:

α = (2,15,19,18,17)(3,23,24,11,12)(4,5,6,21,20)(7,8,9,10,16)(13,29,26,22,14)(25,32,27,28,30),

β = (1,2,3,4,16,10,11,13)(5,15,14,17,9,25,24,28)(6,29,30,31,23,22,8,7)(12,18,32,26,27,21,20,19).

In the CAYLEY program 2 card(s) and card(t) must be greater than 1.

Modifications which have to be made in Theorem 4.1.1 are as follows :

1. If $m \equiv 1 (\mathrm{mod} 8)$, $m \neq 40t + 33$ and $n = 1$ then $F_9 = F'_9$.

2. From relation 3 and using relation 1 it can be easily seen that b^8 commutes with a.

3. If $m \equiv 1 (\mathrm{mod} 8)$, $m \neq 40t + 33$ and $n = 1$ then $b^8 \in Z(F_9) \cap F'_9$.

4. $H_9 = \langle b^8 \rangle$.

5. $K_9 = \langle a, b^2a^{-2}b^{-1}a^{-3}ba^{-2}b^{-3}a^{-1} \rangle$.

6. $L_9 = \langle a \rangle$.

7. Generating pair for SL(2,31) is (a, a^2b^{-2}).

8. In the CAYLEY program 1 card(s) and card(t) must be greater than 3.

Theorem 4.1.10 If ($m \equiv 2 \pmod 9$ and $n = 2$) or ($m \equiv 7 \pmod 9$ and $n = 7$) or ($m \equiv 4 \pmod 9$ and $n = 4$) or ($m \equiv 5 \pmod 9$ and $n = 5$) then

$$F_{10} = \langle a,b \mid a^9 = (ab)^2 = b^{m+9}a^{-n}b^m a^{-n} = 1 \rangle \cong PSL(2,37)$$

and this is an efficient presentation for F_{10}

Case $m \equiv 7 \pmod 9$ and $n = 7$: If $m \equiv 7 \pmod 9$ and $n = 7$ then

$$F_{10} \cong \langle a,b \mid a^9 = (ab)^2 = b^{p+9}a^{-q}b^p a^{-q} = 1 \rangle$$

where $p \equiv 2 \pmod 9$ and $q = 2$.

Proof Let $m \equiv 7 \pmod 9$ and $n = 7$. Then

$$F_{10} = \langle a,b \mid a^9 = (ab)^2 = b^{m+9}a^{-7}b^m a^{-7} = 1 \rangle .$$

Using the map $b \mapsto b^{-1}$, $a \mapsto a^{-1}$ we get

$$F_{10} = \langle a,b \mid a^9 = (ab)^2 = b^{-m-9}a^7 b^{-m} a^7 = 1 \rangle .$$

Now let $-m = p + 9$.

Since $m \equiv 7 \pmod 9$ then $p \equiv 2 \pmod 9$.

Replacing $-m = p + 9$ in F_{10} and using relation 1 the result follows as claimed.

For this case :

The permutation generating pair of degree 38 is:

α = (2,3,22,21,20,19,18,16,17)(4,5,6,7,8,28,27,26,24)(9,14,15,29,36,10,11,12,13)(23,38,31,32,33,34,35,30,37),

β = (1,17,14,7,15,16,11,13,2)(3,12,18,23,22,24,26,5,4)(6,27,33,31,30,8,9,10,29)(19,34,28,35,20,25,21,37,38).

In the CAYLEY program 2 card(s) and card(t) must be greater than 6.

Modifications which have to be made in Theorem 4.1.1 are as follows:

1. If $m \equiv 2 \pmod 9$ and $n = 2$ then $F_{10} = F'_{10}$.

2. From relation 3 and using relation 1 it can be easily seen that b^9 commutes with a^2. The order of a is odd, so b^9 commutes with a.
3. If $m \equiv 2(\mathrm{mod}9)$ and $n = 2$ then $b^9 \in Z(F_{10}) \cap F'_{10}$.
4. $H_{10} = \langle b^9 \rangle$.
5. $K_{10} = \langle a, b^{-5}a^{-6}b^{-2}a^{-4}ba^{-1}b \rangle$.
6. $L_{10} = \langle a \rangle$.
7. Generating pair for SL(2,37) is (a,ab).
8. In the CAYLEY program1 card(s) and card(t) must be greater than 1

Case $m \equiv 5(\mathrm{mod}9)$ and $n = 5$: If $m \equiv 5(\mathrm{mod}9)$ and $n = 5$ then

$$F_{10} \cong \langle a,b \mid a^9 = (ab)^2 = b^{p+9}a^{-q}b^pa^{-q} = 1 \rangle$$

where $p \equiv 4(\mathrm{mod}9)$ and $q = 4$.

Proof Let $m \equiv 5(\mathrm{mod}9)$ and $n = 5$ then

$$F_{10} = \langle a,b \mid a^9 = (ab)^2 = b^{m+9}a^{-5}b^ma^{-5} = 1 \rangle.$$

Using the map $b \mapsto b^{-1}, a \mapsto a^{-1}$ we get

$$F_{10} = \langle a,b \mid a^9 = (ab)^2 = b^{-m-9}a^5b^{-m}a^5 = 1 \rangle.$$

Now let $-m = p + 9$.

Since $m \equiv 5(\mathrm{mod}9)$ then $p \equiv 4(\mathrm{mod}9)$.

Replacing $-m = p + 9$ in F_{10} and using relation 1 the result follows as claimed.

For this case :

The permutation generating pair of degree 38 is:

α = (2,6,17,16,15,14,4,5,3)(7,8,9,10,11,21,22,23,13)(12,31,30,32,24,29,27,19,20)(18,28,25,35,37,33,38,34,26),

β = (1,2,13,12,11,7,3,4,6)(8,10,21,20,19,18,15,32,30)(9,31,23,37,36,35,34,33,22)(14,26,25,24,16,29,28,27,17).

In the CAYLEY program 2 card(s) and card(t) must be greater than 6.

Modifications which have to be made in Theorem 4.1.1 are as follows:

1. If $m \equiv 4 \pmod 9$ and $n = 4$ then $F_{10} = F'_{10}$.
2. From relation 3 and using relation 1 it can be easily seen that b^9 commutes with a^2. The order of a is odd, so b^9 commutes with a.
3. If $m \equiv 4 \pmod 9$ and $n = 4$ then $b^9 \in Z(F_{10}) \cap F'_{10}$.
4. $H_{10} = \langle b^9 \rangle$.
5. $K_{10} = \langle a, b^{-3}a^{-2}b^{-1}ab^{-1}a \rangle$.
6. $L_{10} = \langle a \rangle$.
7. Generating pair for SL(2,37) is (a,ab).
8. In the CAYLEY program 1 card(s) and card(t) must be greater than 1

Theorem 4.1.11 If $m \equiv 16 \pmod{21}$, $m \neq 105t + 100$ and $n = 2$ then

$$F_{11} = \langle a,b \mid a^5 = (ab)^2 = b^{m+21}a^{-n}b^m a^{-n} = 1 \rangle \cong PSL(2,41)$$

and this is an efficient presentation for F_{11}.

The permutation generating pair of degree 42 is:

$\alpha = (1,12,11,10,8)(2,3,26,28,13)(4,5,6,7,27)(9,23,24,34,29)(14,42,30,20,22)$
$(15,19,31,39,41)(16,33,25,36,18)(17,37,35,38,32)$,

$\beta = (1,7,25,24,23,10,22,21,20,19,18,17,16,15,14,11,26,27,8,9,2)(3,29,37,36,6,$
$35,34,33,32,31,30,28,12,13,42,41,40,39,38,5,4)$.

In the CAYLEY program 2 card(s) and card(t) must be greater than 1.

Modifications which have to be made in Theorem 4.1.1 are as follows:

1. If $m \equiv 16 \pmod{21}$, $m \neq 105t + 100$ and $n = 2$ then $F_{11} = F'_{11}$.
2. From relation 3 and using relation 1 it can be easily seen that b^{21} commutes with a^2. The order of a is odd, so b^{21} commutes with a.
3. If $m \equiv 16 \pmod{21}$, $m \neq 105t + 100$ and $n = 2$ then $b^{21} \in Z(F_{11}) \cap F'_{11}$.

4. $H_{11} = \langle b^{21} \rangle$.

5. $K_{11} = \langle b^{-3}a, ba^{-3}ba^{-1}b \rangle$.

6. $L_{11} = \langle b \rangle$.

7. Generating pair for SL(2,41) is (a,ab).

8. In the CAYLEY program 1 card(s) and card(t) must be greater than 5

Theorem 4.1.12 If $m \equiv 2(\text{mod}11)$, $m \neq 77t + 24$ and $n = 2$ then

$$F_{12} = \langle a,b \mid a^7 = (ab)^2 = b^{m+11}a^{-n}b^m a^{-n} = 1 \rangle \cong PSL(2,43)$$

and this is an efficient presentation for F_{12}.

The permutation generating pair of degree 44 is:

α = (2,17,26,25,24,23,22)(3,4,5,6,7,16,18)(8,9,32,34,15,39,36)(10,11,12,13,21, 35, 40)(14,42,43,41,37,33,38)(19,31,30,27,28,29,20),

β = (1,2,3,21,20,19,13,14,15,16,17)(4,22,32,36,35,18,34,23,33,25,30) (5,31,29,41,40,39,38,24,37,28,11)(6,10,43,44,42,12,27,26,7,8,9).

In the CAYLEY program 2 card(s) and card(t) must be greater than 1.

Modifications which have to be made in Theorem 4.1.1 are as follows:

1. If $m \equiv 2(\text{mod}11)$, $m \neq 77t + 24$ and $n = 2$ then $F_{12} = F'_{12}$.

2. From relation 3 and using relation 1 it can be easily seen that b^{11} commutes with a^2. The order of a is odd, so b^{11} commutes with a.

3. If $m \equiv 2(\text{mod}11)$, $m \neq 77t + 24$ and $n = 2$ then $b^{11} \in Z(F_{12}) \cap F'_{12}$.

4. $H_{12} = \langle b^{11} \rangle$.

5. $K_{12} = \langle a, b^{-5}a^{-3}b^{-4}a^{-4}b^{-2}a^{-1} \rangle$.

6. $L_{12} = \langle a \rangle$.

7. Generating pair for SL(2,43) is (a,ab).

8. In the CAYLEY program 1 card(s) and card(t) must be greater than 1.

Theorem 4.1.13 If $m \equiv 3(\text{mod}9)$, $m \neq 117t + 48$ and $n = 6$ then

$$F_{13} = \langle\, a,b \mid a^{13} = (ab)^2 = b^{m+9}a^{-n}b^{m}a^{-n} = 1 \,\rangle \cong PSL(2,53)$$

and this is an efficient presentation for F_{13}.

The permutation generating pair of degree 54 is:

α = (2,22,34,16,33,32,23,31,8,30,29,17,28)(3,10,11,35,36,37,24,39,38,7,6,5,4)(9,46,43,44,20,19,12,18,26,27,25,40,47)(13,14,15,50,42,51,52,41,53,49,21,45,48),

β = (1,2,3,25,24,23,20,21,22)(4,45,44,43,26,29,42,41,40)(5,35,19,32,15,16,17,18,48)(6,7,8,9,10,28,34,49,36)(11,47,52,54,51,30,38,13,12)(14,39,27,46,31,37,53,50,33).

In the CAYLEY program 2 card(s) and card(t) must be greater than 4.

Modifications which have to be made in Theorem 4.1.1 are as follows:

1. If $m \equiv 3(\text{mod}9)$, $m \neq 117t + 48$ and $n = 6$ then $F_{13} = F'_{13}$.
2. From relation 3 and using relation 1 it can be easily seen that b^9 commutes with a^6. The order of a is 13, so b^{11} commutes with a.
3. If $m \equiv 3(\text{mod}9)$, $m \neq 117t + 48$ and $n = 6$ then $b^{11} \in Z(F_{13}) \cap F'_{13}$.
4. $H_{13} = \langle b^9 \rangle$.
5. $K_{13} = \langle a, b^{-3}a^3b^{-3}a^{-2}b^2a^{-1}b^{-3}a^4b^{-2}a^2 \rangle$.
6. $L_{13} = \langle a \rangle$.
7. Generating pair for SL(2,53) is (a,ab).
8. In the CAYLEY program 1 card(s) and card(t) must be greater than 4.

Theorem 4.1.14 If $m \equiv 9(\text{mod}15)$ and $n = 3$ then

$$F_{14} = \langle\, a,b \mid a^5 = (ab)^2 = b^{m+15}a^{-n}b^{m}a^{-n} = 1 \,\rangle \cong PSL(2,59)$$

and this is an efficient presentation for F_{14}.

The permutation generating pair of degree 60 is:

α = (1,9,24,32,55)(2,30,16,49,38)(3,43,41,6,34)(4,29,19,48,7)(5,11,14,42,15)
(8,50,36,13,46)(10,25,56,60,54)(12,33,26,35,47)(17,52,45,31,18)(20,23,21,59,39)
(22,53,27,40,58)(28,51,37,44,57),

β = (1,15,59,28,31,39,42,58,7,25,53,9,6,17,34)(2,37,32,50,5,55,51,21,60,46,54,
23,49,43,26)(3,18,57,30,33,19,52,41,16,44,38,20,45,29,35)(4,40,11,8,56,48,12,
36,24,22,14,27,10,13,47).

In the CAYLEY program 2 card(s) and card(t) must be greater than 2.

Modifications which have to be made in Theorem 4.1.1 are as follows:

1. If $m \equiv 9 (\mathrm{mod} 15)$ and $n = 3$ then $F_{14} = F'_{14}$.
2. From relation 3 and using relation 1 it can be easily seen that b^{15} commutes with a^3. The order of a is 5, so b^{15} commutes with a.
3. If $m \equiv 9 (\mathrm{mod} 15)$ and $n = 3$ then $b^{11} \in Z(F_{14}) \cap F'_{14}$.
4. $H_{14} = \langle b^{15} \rangle$.
5. $K_{14} = \langle (a^{-2}b^{-2})^2a^{-2}b, b^{-1}a^2b^3a^{-2}b^{-9}ab^{-5}ab^{-4}a^{-1} \rangle$.
6. $L_{14} = \langle b \rangle$.
7. Generating pair for SL(2,59) is (a,ab).
8. In the CAYLEY program 1 card(s) and card(t) must be greater than 8.

Theorem 4.1.15 If $m \equiv 1 (\mathrm{mod} 3)$, $m \neq 39t + 19$ and $n = 7$ then

$$F_{15} = \langle a,b \mid a^{13} = (a^2b^{-1})^2 = b^{m+3}a^{-n}(b^ma^{-n})^3 = 1 \rangle \cong PSL(2,79)$$

and this is an efficient presentation for F_{15}.

The permutation generating pair of degree 80 is:

α = (2,3,39,38,21,22,23,24,25,26,27,28,29)(4,5,6,7,8,37,36,35,34,33,32,31,30)
(9,10,76,77,78,79,70,69,68,67,66,65,64)(11,12,59,75,50,49,48,47,46,45,44,43,

60)(13,14,15,16,17,18,61,62,63,72,20,19,71)(40,41,42,58,57,56,55,54,53,52,51, 73,74),

β = (1,27,25)(2,8,34)(3,62,50)(4,78,52)(5,66,49)(6,39,76)(7,60,68)(9,21,46) (10,11,58)(12,54,70)(13,69,24)(14,40,57)(15,63,79)(16,31,73)(17,80,61) (18,38,33)(19,26,74)(20,43,77)(22,67,35)(23,29,48)(28,32,51)(30,47,53) (37,65,44)(41,59,72)(45,71,55)(56,75,64).

In the CAYLEY program 2 card(s) and card(t) must be greater than 1.

Modifications which have to be made in Theorem 4.1.1 are as follows:

1. If $m \equiv 1 (\mathrm{mod} 3)$, $m \neq 39t + 19$ and $n = 7$ then $F_{15} = F'_{15}$.
2. From relation 3 and using relation 1 it can be easily seen that b^3 commutes with a^7. The order of a is 13, so b^3 commutes with a.
3. If $m \equiv 1 (\mathrm{mod} 3)$, $m \neq 39t + 19$ and $n = 7$ then $b^3 \in Z(F_{15}) \cap F'_{15}$.
4. $H_{15} = \langle b^3 \rangle$.
5. $K_{15} = \langle a, ba^7b^{-1}a^{-1}ba^5b^{-1}ababa^4bab \rangle$.
6. $L_{15} = \langle a \rangle$.
7. Generating pair for SL(2,79) is (a, a^2b^{-1}).
8. In the CAYLEY program 1 card(s) and card(t) must be greater than 3.

Theorem 4.1.16 If $m \equiv 7 (\mathrm{mod} 21)$ and $n = 3$ then

$$F_{16} = \langle a,b \mid a^7 = (ab)^2 = b^{m+21}a^{-n}b^m a^{-n} = 1 \rangle \cong PSL(2,83)$$

and this is an efficient presentation for F_{16}.

The permutation generating pair of degree 84 is:

α = (1,4,7,12,23,16,58)(2,53,74,43,54,44,3)(5,10,62,73,11,46,63)(6,32,40,25,78, 18,9)(8,72,41,31,48,76,66)(13,20,65,81,27,80,24)(14,17,21,47,15,60,37)(19,67, 64,55,42,71,38)(22,50,84,29,75,68,59)(26,39,70,49,45,28,77)(30,83,34,82,35,79, 69)(33,36,57,56,52,51,61),

β = (1,21,23,24,25,26,27,28,29,30,31,32,33,34,35,36,6,5,4,3,2)(7,63,66,67,68, 65,48,69,70,71,72,18,15,14,13,12,11,10,9,8,46)(16,17,47,58,59,19,60,78,80,77, 81,75,45,57,82,61,62,73,51,74,22)(20,37,38,39,40,41,42,43,52,54,55,50,53,44, 56,49,79,83,84,64,76).

In the CAYLEY program 2 card(s) and card(t) must be greater than 3.

Modifications which have to be made in Theorem 4.1.1 are as follows:

1. If $m \equiv 7 (\mathrm{mod} 21)$ and $n = 3$ then $F_{16} = F'_{16}$.
2. From relation 3 and using relation 1 it can be easily seen that b^{21} commutes with a^3. The order of a is 7, so b^{21} commutes with a.
3. If $m \equiv 7 (\mathrm{mod} 21)$ and $n = 3$ then $b^{21} \in Z(F_{16}) \cap F'_{16}$.
4. $H_{16} = \langle b^{21} \rangle$.
5. $K_{16} = \langle a^2b^{-7}a^{-2}b^{-5}, a^{-1}ba^{-1} \rangle$.
6. $L_{16} = \langle b \rangle$.
7. Generating pair for SL(2,83) is (a,ab).
8. In the CAYLEY program 1 card(s) and card(t) must be greater than 3.

Theorem 4.1.17 If $m \equiv 3 (\mathrm{mod} 9)$, $m \neq 99t + 84$ and $n = 5$ then

$$F_{17} = \langle a,b \mid a^{11} = (ab)^2 = b^{m+9}a^{-n}b^m a^{-n} = 1 \rangle \cong PSL(2,89)$$

and this is an efficient presentation for F_{17}.

The permutation generating pair of degree 90 is:

α = (2,23,24,25,26,27,28,29,30,31,19)(3,35,36,37,38,39,40,32,41,42,43) (4,22,21,15,16,17,51,5,6,7,34)(8,13,70,53,33,71,72,73,63,61,74) (9,69,68,80,81,60,44,62,10,11,12)(14,59,20,50,49,48,47,46,67,87,89) (18,58,57,56,45,55,90,65,83,82,52)(54,84,86,66,79,78,77,64,76,75,85),

β = (1,19,18,17,20,5,4,3,2)(6,59,21,53,54,55,60,29,61)(7,63,25,64,65,66,10,9,8)(11,86,48,87,15,14,13,12,88)(16,67,56,68,26,73,36,82,50)(22,51,52,35,34,74,28,32,33)(23,43,71,40,44,45,46,58,31)(24,30,81,78,37,72,42,75,76)(27,69,62,39,38,79,90,85,41)(47,84,70,89,49,83,77,80,57).

In the CAYLEY program 2 card(s) and card(t) must be greater than 1.

Modifications which have to be made in Theorem 4.1.1 are as follows:

1. If $m \equiv 3 (\mathrm{mod} 9)$, $m \neq 99t + 84$ and $n = 5$ $F_{17} = F'_{17}$.
2. From relation 3 and using relation 1 it can be easily seen that b^9 commutes with a^5. The order of a is 11, so b^9 commutes with a.
3. If $m \equiv 3 (\mathrm{mod} 9)$, $m \neq 99t + 84$ and $n = 5$ $b^9 \in Z(F_{17}) \cap F'_{17}$.
4. $H_{17} = \langle b^9 \rangle$.
5. $K_{17} = \langle a, ba^7b^{-1}a^{-1}ba^5b^{-1}ababa^4bab \rangle$.
6. $L_{17} = \langle a \rangle$.
7. Generating pair for SL(2,89) is (a,ab).
8. In the CAYLEY program 1 card(s) and card(t) must be greater than 1.

Theorem 4.1.18 If $m \equiv 2 (\mathrm{mod} 11)$, $m \neq 99t + 24$ and $n = 2$ then

$$F_{18} = \langle a,b \mid a^9 = (ab)^2 = b^{m+11}a^{-n}b^m a^{-n} = 1 \rangle \cong PSL(2,109)$$

and this is an efficient presentation for F_{18}.

The permutation generating pair of degree 110 is:

α = (1,6,26,25,24,9,23,22,21)(2,7,8,43,42,41,40,39,3)(4,5,20,57,56,55,30,31,32)(10,78,77,76,74,75,11,72,71)(12,13,14,15,16,17,18,58,50)(19,67,37,63,53,91,51,94,93)(27,100,70,44,99,33,60,61,28)(29,82,90,54,85,86,69,59,34)(35,36,66,65,84,47,48,49,92)(38,87,79,62,73,101,105,45,46)(52,64,68,97,98,103,95,96,102)(81,89,83,106,104,110,109,108,107),

β = (1,2,32,38,37,36,35,25,34,33,27)(3,50,49,54,53,52,51,30,29,26,4)(5,6,28,65,64,63,46,62,18,19,20)(7,21,14,56,69,68,66,67,17,47,23)(8,9,10,11,12,39,43,70,73,45,44)(13,75,80,74,15,22,84,61,83,82,55)(16,76,87,31,91,90,89,88,81,85,48)(24,92,58,79,77,41,72,71,42,40,78)(57,93,104,98,95,109,108,103,106,60,59)(86,107,110,94,102,101,100,99,105,96,97).

In the CAYLEY program 2 card(s) and card(t) must be greater than 1.

Modifications which have to be made in Theorem 4.1.1 are as follows:

1. If $m \equiv 2 (\mathrm{mod} 11)$, $m \neq 99t + 24$ and $n = 2$ then $F_{18} = F'_{18}$.
2. From relation 3 and using relation 1 it can be easily seen that b^{11} commutes with a^2. The order of a is odd, so b^{11} commutes with a.
3. If $m \equiv 2 (\mathrm{mod} 11)$, $m \neq 99t + 24$ and $n = 2$ then $b^{11} \in Z(F_{18}) \cap F'_{18}$.
4. $H_{18} = \langle b^{11} \rangle$.
5. $K_{18} = \langle ab^{-1}a^{-1}bab^{-1}, ba^{-1}b^{-1}ab^{-3}a^{-6}b^{-4}a^{-2}b^{-1} \rangle$.
6. $L_{18} = \langle b \rangle$.
7. Generating pair for SL(2,109) is (a,ab).
8. In the CAYLEY program 1 card(s) and card(t) must be greater than 4.

Theorem 4.1.19 If $m \equiv 7 (\mathrm{mod} 23)$, $m \neq 115t + 99$ and $n = 2$ then

$$F_{19} = \langle a,b \mid a^5 = (ab)^2 = b^{m+23}a^{-n}b^{m}a^{-n} = 1 \rangle \cong PSL(2,139)$$

and this is an efficient presentation for F_{19}.

The permutation generating pair of degree 140 is:

α = (1,42,25,3,2)(4,24,76,39,46)(5,47,81,7,6)(8,82,114,72,97)(9,90,120,65,59)(10,60,87,122,48)(11,49,36,70,130)(12,132,131,94,43)(13,14,15,92,104)(16,98,80,40,77)(17,78,127,136,83)(18,84,117,67,62)(19,51,125,37,50)(20,21,22,23,95)(26,27,28,55,138)(29,63,75,44,102)(30,103,93,133,71)

(31,32,33,34,41)(35,115,109,123,69)(38,126,79,101,45)(52,86,61,68,124)

(53,112,108,116,85)(54,135,128,129,111)(56,139,57,106,137)(58,66,99,100,105)

(64,121,88,89,74)(73,110,134,91,96)(107,113,118,140,119),

β = (1,27,139,138,135,112,86,125,18,68,87,9,66,106,22,19,36,10,81,4,38,102,
25)(2,24,47,15,12,95,76,3,44,64,110,97,33,30,130,35,124,62,99,59,121,75,28)
(5,98,78,84,51,21,96,120,8,73,20,43,133,104,41,115,70,50,126,46,77,31,92)
(6,82,90,60,49,132,14,105,118,108,54,134,74,123,116,113,137,58,13,93,29,45,
80)(7,48,88,65,91,111,131,11,71,94,129,136,79,37,52,69,89,122,61,53,117,17,
114)(16,39,23,57,26,56,107,140,100,67,85,109,34,72,83,128,55,63,103,32,40,
101,127).

In the CAYLEY program 2 card(s) and card(t) must be greater than 4.

Modifications which have to be made in Theorem 4.1.1 are as follows:

1. If $m \equiv 7(\mathrm{mod}23)$, $m \neq 115t+99$ and $n = 2$ then $F_{19} = F'_{19}$.
2. From relation 3 and using relation 1 it can be easily seen that b^{23} commutes with a^2. The order of a is odd, so b^{23} commutes with a.
3. If $m \equiv 7(\mathrm{mod}23)$, $m \neq 115t+99$ and $n = 2$ then $b^{23} \in Z(F_{19}) \cap F'_{19}$.
4. $H_{19} = \langle b^{23} \rangle$.
5. $K_{19} = \langle b, a^{-3}b^{-2}a^{-3}b^{-6}a^{-2}b^{-6}a^2b^{-2}a^2, a^{-2}b^2a^{-1}b^{-9}a^{-3}b^{-3}a^{-2}b^{-1}a \rangle$.
6. $L_{19} = \langle b \rangle$.
7. Generating pair for SL(2,139) is (a,ab).
8. In the CAYLEY program 1 card(s) and card(t) must be greater than 5.

Theorem 4.1.20 If $m \equiv 2(\mathrm{mod}5)$, $m \neq 95t+77$ and $n = 7$ then

$$F_{20} = \langle a,b \mid a^{19} = (ab)^2 = b^{m+5}a^{-n}b^m a^{-n} = 1 \rangle \cong PSL(2,229)$$

and this is an efficient presentation for F_{20}.

The permutation generating pair of degree 230 is:

α = (1,9,97,42,149,178,77,43,148,67,28,86,24,191,82,172,12,135,44)(2,70,196, 180,145,100,179,53,200,27,125,74,26,14,46,223,138,115,133)(3,33,130,162,62, 105,169,156,158,181,126,189,214,194,11,17,59,141,168)(4,66,204,45,117,122, 72,107,147,102,94,15,213,93,188,134,166,106,10)(5,85,230,120,227,210,20,123,173, 25,226,174,30,197,203,212,154,140,187)(6,87,150,31,128,124,224,88,57, 190,118,109,116,219,68,103,171,228,41)(7,96,113,209,22,199,112,47,119,205, 21,153,220,218,51,78,225,35,75)(8,58,81,32,95,39,90,131,80,50,89,99,108,184, 36,155,215,157,151)(13,60,139,177,111,69,63,49,52,211,29,165,143,170,159,98, 167,16,65)(18,76,61,182,201,73,132,54,110,161,34,38,121,192,84,144,137,55, 48)(19,136,229,64,37,160,92,163,198,183,129,71,176,185,83,114,164,193,221) (23,79,195,146,222,56,202,175,207,208,40,186,91,206,142,152,216,127,217),

β = (1,2,37,59,58)(3,14,18,81,17)(4,16,15,32,48)(5,88,96,95,94)(6,97,106,39,7) (8,38,65,10,9)(11,12,13,34,33)(19,47,46,168,82)(20,60,172,141,64)(21,143,178, 130,161)(22,179,182,162,149)(23,24,25,62,61)(26,150,80,79,76)(27,77,165,195, 131)(28,132,159,146,29)(30,147,99,98,73)(31,74,213,167,89)(35,49,127,135,194) (36,193,70,44,216)(40,114,199,42,41)(43,200,196,164,208)(45,71,119,221,184) (50,87,75,214,128)(51,91,124,189,188)(52,225,134,126,92)(53,209,163,181,180) (54,67,145,158,157)(55,121,151,156,66)(56,176,204,169,120)(57,187,227,105, 173)(63,142,136,191,217)(68,153,110,215,144)(69,192,137,155,152)(72,116,117, 108,107)(78,93,125,90,166)(83,183,171,223,112)(84,111,154,138,103)(85,102, 174,115,212)(86,211,160,133,226)(100,148,207,197,201)(101,202,230,203,175) (104,139,210,140,177)(109,118,220,219,122)(113,224,186,228,198)(123,229,206, 218,190) (129,185,222,170,205).

In the CAYLEY program 2 card(s) and card(t) must be greater than 1.

Modifications which have to be made in Theorem 4.1.1 are as follows:

1. If $m \equiv 2 (\mathrm{mod} 5)$, $m \neq 95t + 77$ and $n = 7$ then $F_{20} = F'_{20}$.
2. From relation 3 and using relation 1 it can be easily seen that b^5 commutes with a^7. The order of a is 19, so b^5 commutes with a.
3. If $m \equiv 2 (\mathrm{mod} 5)$, $m \neq 95t + 77$ and $n = 7$ then $b^5 \in Z(F_{20}) \cap F'_{20}$.
4. $H_{20} = \langle b^5 \rangle$.
5. $K_{20} = \langle a^{-2}b^{-2}ab^2a^{-1}ba^{-4}b^{-1}, a^3b^{-3}a^{-1}ba^{-2}ba^{-4}b^{-1}, ba^4b^{-1}a^2b^3a^{-1}b^2a^{-1}b^2a^{-4}b^2 a^{-6}ba^{-5}ba^{-3}b^{-2}a^{-4}b^{-1}a \rangle$.
6. $L_{20} = \langle a \rangle$.
7. Generating pair for SL(2,229) is (a,ab).
8. In the CAYLEY program 1 card(s) and card(t) must be greater than 1.

4.2 DIFFERENT EFFICIENT PRESENTATIONS FOR THE GROUPS $PSL(2,p^n)$ where $p^n \in \{9,25,27,49,169\}$:

As we mentioned in Chapter 4.1 considerable effort has, over the years, been put into finding presentations for the groups PSL(2,p), p prime, with optimal defining relations. But on the corresponding problem of finding presentations with optimal defining relations for the groups $PSL(2,p^n)$, p prime, $n \geq 2$ this is not the case, for except for a few cases of particular values of p and n, we can find nothing in the literature. There are general results, although they do not give efficient presentations.

Presentations of $PSL(2,p^n)$, p prime, $n \geq 2$ with the number of defining relations increasing with n are given by Bussey [5], Todd [38], Sinkov [35] and Beetham [3]. We can also find work done on $PSL(2,p^n)$ by P.D.Williams in [42].

Recently C.M.Campbell, E.F.Robertson and P.D.Williams in [12] have given presentations for the groups $PSL(2,p^n)$, p prime, which show that the deficiency of these groups is bounded. In the same paper in particular, they have given efficient presentations for $PSL(2,3^4)$, $PSL(2,5^3)$, $PSL(2,11^2)$, $PSL(2,13^2)$ and $PSL(2,19^2)$. Also C.M.Campbell and E.F.Robertson have given an efficient presentation for $PSL(2,3^2)$. Efficient presentations of $PSL(2,3^3)$, $PSL(2,5^2)$ and $PSL(2,7^2)$ are also given in [9].

In this section we study groups of the following types

(i) $\langle a,b \mid a^k = (ab)^2 = b^{m+s}a^{-n}(b^m a^{-n})^4 = 1 \rangle$

(ii) $\langle a,b \mid a^k = (ab)^2 = b^{m+s}a^{-n}b^m a^{-n} = 1 \rangle$

In this section we are going to show that for certain values of k,s,m,n these groups are efficient and isomorphic to the groups $PSL(2,p^n)$.

Theorem 4.2.1 If $m \equiv 1(\mathrm{mod}24)$ and $n = 1$ then

$$F_1 = \langle a,b \mid a^5 = (ab)^2 = b^{m+4}a^{-n}(b^m a^{-n})^4 = 1 \rangle \cong PSL(2,9)$$

and this is an efficient presentation for F_1.

Proof: If $m \equiv 1(\mathrm{mod}24)$ and $n = 1$ then $F_1 = F'_1$ and consequently $b^4 \in F_1 = F'_1$. From relation 3 and using relation 1 it can be easily seen that b^4 commutes with a, so $b^4 \in Z(F_1)$. Since $b^4 \in F'_1$ it can be noticed that $b^4 \in Z(F_1) \cap F'_1$.

Now consider the homomorphic image of F_1 by $H_i = \langle b^4 \rangle$ in other words

$$F_1/\langle b^4 \rangle = \langle a,b \mid a^5 = (ab)^2 = (ba^{-1})^5 = b^4 = 1 \rangle.$$

Now take $K_i = \langle a^{-1}ba^{-1}, b^{-1}ab^{-2}a \rangle < F_1/\langle b^4 \rangle$

Using CAYLEY program 1 it can be seen that permutation representation of $F_1/\langle b^4 \rangle$ on the cosets of K_i is isomorphic to PSL(2,9) group.

Using TC on $L_i = \langle a \rangle < F_1/\langle b^4 \rangle$ it can be verified that the order of $F_1/\langle b^4 \rangle$ is equal to the order of the group PSL(2,9). Hence $F_1/\langle b^4 \rangle \cong PSL(2,9)$.

Since $b^5 \in Z(F_1) \cap F'_1$ it can be deduced that $\langle b^4 \rangle \leq M(PSL(2,9)) = C_6$. This means $\mid \langle b^4 \rangle \mid = 1$ or 2 or 3 or 6. In any case $b^{24} = 1$ holds in F_1. Adding this relation into the group and using the fact that $m \equiv 1(\mathrm{mod}24)$ and $b^4 \in Z(F_1)$ the following can be deduced

$$F_1 = \langle a,b \mid a^5 = (ab)^2 = b^5a^{-1}(ba^{-1})^4 = b^{24} = 1 \rangle$$

Again using TC on $L_i = \langle a \rangle$ it can be seen that $\mid F_1 \mid = \mid F_1/\langle b^4 \rangle \mid$, therefore $F_1 \cong PSL(2,9)$.

F_1 has 2 generators, 3 relations and the Schur multiplier of F_1 is C_6 so this is an efficient presentation for F_1.

Theorem 4.2.2 If $m \equiv 2 (\mathrm{mod} 5)$, $m \neq 65t + 12$ and $n = 5$ then

$$F_2 = \langle a,b \mid a^{13} = (ab)^2 = b^{m+5}a^{-n}b^m a^{-n} = 1 \rangle \cong PSL(2,25)$$

and this is an efficient presentation for F_2.

Proof: If $m \equiv 2(\mathrm{mod}5)$, $m \neq 65t + 12$ and $n = 5$ then $F_2 = F'_2$ and consequently $b^5 \in F_2 = F'_2$. From relation 3 and using relation 1 it can be easily seen that b^5 commutes with a^5. Using this fact and the fact that 5 is coprime to 13, the order of a, b^5 commutes with a therefore $b^5 \in Z(F_2)$. Since $b^5 \in F'_2$ it can be noticed that $b^5 \in Z(F_2) \cap F'_2$.

Now consider the homomorphic image of F_2 by $H_i = \langle b^5 \rangle$ in other words

$$F_2/\langle b^5 \rangle = \langle a,b \mid a^{13} = (ab)^2 = (b^2a^{-5})^2 = b^5 = 1 \rangle$$

Now take $K_i = \langle a^{-1}ba^{-1}, a^{-9}b^{-2}a^{-4}b^{-1}a^3b^{-2}a \rangle < F_2/\langle b^5 \rangle$

Using the CAYLEY program 1 it can be seen that the permutation representation of $F_2/\langle b^5 \rangle$ on the cosets of K_i is isomorphic to the group PSL(2,25).

Using TC on $L_i = \langle a \rangle < F_2/\langle b^5 \rangle$ it can be verified that order of $F_2/\langle b^5 \rangle$ is equal to the order of the group PSL(2,25). Hence $F_2/\langle b^5 \rangle \cong PSL(2,25)$.

Since $b^5 \in Z(F_2) \cap F'_2$ it can be deduced that $\langle b^5 \rangle \leq M(PSL(2,25)) = C_2$. This means $| \langle b^5 \rangle | = 1$ or 2. i.e.

$F_2 \cong$ PSL(2,25) or F_2 is isomorphic to its covering group, SL(2,25).

In either case $b^{10} = 1$ holds in F_2.

Adding this relation to the group and using the fact that $m \equiv 2(\mathrm{mod}5)$, $m \neq 65t + 12$ and $b^5 \in Z(F_2)$, the following can be deduced:

$$F_2 = \langle a,b \mid a^{13} = (ab)^2 = (b^2a^{-5})^2 = b^{10} = 1 \rangle$$

Again using TC on $L_i = \langle a \rangle$ it can be seen that $| F_2 | = | F_2/\langle b^5 \rangle |$.

Therefore $F_2 \cong PSL(2,25)$.

F_2 has 2 generators, 3 relations and the Schur multiplier of F_2 is C_2 so this is an efficient presentation for F_2.

In the following three theorems full proofs are essentially the same as the proof of Theorem 4.2.2 with modifications. In every case instead of full proofs only the modifications will be given which have to be made in the proof of Theorem 4.2.2 and in the CAYLEY program 1, in order to obtain the full proof. These modifications are:

1. The conditions which makes F_i perfect.
2. The proof that b^r commutes with a.
3. The element b^r which is in $Z(F_i) \cap F'_i$.
4. The subgroup H_i which is going to be used to construct the homomorphic image $F_i/\langle b^r \rangle$.
5. The subgroup K_i which is going to be used to construct the permutation representation of $F_i/\langle b^r \rangle$ on the cosets of K_i.
6. The subgroup L_i which is going to be used to enumerate $F_i/\langle b^r \rangle$ and $F_i/\langle b^{2r} \rangle$.
7. The conditions for card(s) and card(t) in the CAYLEY program 1.

Additionally it can be pointed out that, actually K_i is a maximal subgroup of F_i with minimal index. Therefore with the CAYLEY program 2, a permutation representation has been obtained for these F_i groups. In every case we give these permutation generating pairs.

For Theorem 4.2.1 and Theorem 4.2.2 the permutation generating pairs are respectively of degree 6 and 26 as follows :

$\alpha = (1,2,6,5,3)$,

$\beta = (1,3,2,4)(5,6)$.

$\alpha = (1,2,22,21,14,15,16,17,13,18,19,20,3)(4,23,26,24,8,9,10,11,12,25,7,6,5)$,

$\beta = (1,9,25,19,23)(2,4,5,10,3)(6,16,26,18,11)(7,8,24,15,17)(12,13,14,22,20)$.

In the CAYLEY program 2, for both theorems, card(s) and card(t) must be greater than 1.

Theorem 4.2.3 If $m \equiv 3(\mathrm{mod}13)$, $m \neq 91t + 29$ and $n = 3$ then

$$F_3 = \langle\, a,b \mid a^7 = (ab)^2 = b^{m+13}a^{-n}b^m a^{-n} = 1 \,\rangle \cong PSL(2,27)$$

and this is an efficient presentation for F_3.

The permutation generating pair of degree 28 is:

$\alpha = (1,2,4,5,7,6,3)(8,21,20,19,14,17,9)(10,11,12,13,18,15,16)$
$(22,26,28,27,25,23,24)$,

$\beta = (1,15,14,13,10,4,6,5,16,3,2,9,8)(11,12,23,22,20,27,26,25,21,17,18,19,24)$.

In the CAYLEY program card(s) and card(t) must be greater than 1.

Modifications which have to be made in Theorem 4.2.2 are as follows:

1. If $m \equiv 3(\mathrm{mod}13)$, $m \neq 91t + 29$ and $n = 3$ then $F_3 = F_3'$.
2. From relation 3 and using relation 1 it can be easily seen that b^{13} commutes with a^3. Using relation 1 it can be seen that b^{13} commutes with a.
3. If $m \equiv 3(\mathrm{mod}13)$, $m \neq 91t + 29$ and $n = 3$ then $b^{13} \in Z(F_3) \cap F_3'$.
4. $H_3 = \langle\, b^{13} \,\rangle$.
5. $K_3 = \langle\, a^{-1}ba^{-1}, a^{-3}b^{-3}aba^{-2}b^{-1}a \,\rangle$.
6. $L_3 = \langle\, b \,\rangle$.
7. In the CAYLEY program 1 card(s) and card(t) must be greater than 1.

Theorem 4.2.4 If $m \equiv 14(\mathrm{mod}25)$ and $n = 2$ then

$$F_4 = \langle\, a,b \mid a^5 = (ab)^2 = b^{m+25}a^{-n}b^m a^{-n} = 1 \,\rangle \cong PSL(2,49)$$

and this is an efficient presentation for F_4.

The permutation generating pair of degree 50 is:

α = (1,2,4,5,3)(6,13,39,42,43)(7,14,15,9,8)(10,17,38,12,11)(16,26,41,19,18)(20,29,49,47,25)(21,32,44,30,28)(22,37,40,27,31)(23,24,48,33,36)(34,46,50,45,35),

β = (1,37,36,35,34,33,32,31,30,29,28,27,26,11,14,43,42,41,40,3,2,19,39,38,18)(4,5,6,7,50,49,44,48,47,46,45,8,25,24,23,22,21,20,9,10,16,17,15,12,13).

In the CAYLEY program 2 card(s) and card(t) must be greater than 1.

Modifications which have to be made in Theorem 4.2.2 are as follows:

1. If $m \equiv 14(\mathrm{mod}25)$ and $n \equiv 2(\mathrm{mod}5)$ then $F_4 = F'_4$.
2. From relation 3 and using relation 1 it can be easily seen that b^{25} commutes with a^2. The order of a is odd, so b^{25} commutes with a.
3. If $m \equiv 14(\mathrm{mod}25)$ and $n = 2$ then $b^{25} \in Z(F_4) \cap F'_4$.
4. $H_4 = \langle b^{25} \rangle$.
5. $K_4 = \langle a^{-1}ba^{-1}, a^2b^{-7}a^{-3}b^{-5}a^2b^{-1}a^2b^{-2}a^{-3} \rangle$.
6. $L_4 = \langle b \rangle$.
7. In the CAYLEY program 1 card(s) and card(t) must be greater than 1.

Theorem 4.2.5 If $m \equiv 2(\mathrm{mod}5)$, $m \neq 85t + 22$ and $n = 13$ then

$$F_5 = \langle\, a,b \mid a^{17} = (ab)^2 = b^{m+5}a^{-n}b^m a^{-n} = 1 \,\rangle \cong PSL(2,169)$$

and this is an efficient presentation for F_5.

The permutation generating pair of degree 170 is:

α = (1,14,47,46,45,44,43,42,41,40,39,38,37,36,4,3,2)(5,49,96,95,94,93,92,91, 85,90,89,88,87,86,8,7,6)(9,15,16,17,125,124,123,122,121,120,119,118,117,116, 26,11,10)(12,31,30,80,78,35,81,82,25,24,23,22,21,83,84,33,32)(13,48,145,144, 143,142,141,129,140,139,138,137,136,135,134,100,132)(18,19,20,127,63,128, 101,102,103,104,105,106,107,29,28,108,109)(27,126,168,157,156,167,131,158, 159,166,165,112,164,163,162,161,160)(34,57,58,59,60,61,62,97,98,99,50,51,52,53,5 4,55,56)(64,65,66,67,68,69,70,71,72,73,74,75,76,77,113,114,115)(79,153, 154,155,169,133,170,130,146,147,148,110,111,149,150,151,152),

β = (1,6,62,52,48)(2,78,77,4,5)(3,76,33,34,35)(7,110,20,21,97)(8,9,68,102,111) (10,32,75,130,69)(11,123,14,13,12)(15,86,96,99,25)(16,82,135,143,41) (17,42,155,156,18)(19,157,119,133,83)(22,158,45,24,98)(23,46,85,39,159) (26,27,28,92,122)(29,30,79,38,91)(31,132,128,115,153)(36,127,148,50,49) (37,152,139,64,63)(40,144,161,150,166)(43,104,57,84,169)(44,131,118,168,105) (47,124,53,61,90)(51,147,108,160,145)(54,125,109,146,74)(55,73,164,71,94) (56,95,87,134,81)(58,103,67,142,136)(59,137,138,151,162)(60,163,72,112,89) (65,140,113,80,107)(66,106,126,116,141)(70,170,120,121,93)(88,165,149,101, 100)(114,129,117,167,154).

In the CAYLEY program 2 card(s) and card(t) must be greater than 1.

Modifications which have to be made in Theorem 4.2.2 are as follows:

1. If $m \equiv 2 (\mathrm{mod} 5)$, $m \neq 85t + 22$ and $n = 13$ then $F_5 = F'_5$.
2. From relation 3 and using relation 1 it can be easily seen that b^5 commutes with a^{13}. Using relation 1 it can be seen that b^5 commutes with a.
3. If $m \equiv 2 (\mathrm{mod} 5)$, $m \neq 85t + 22$ and $n = 13$ then $b^5 \in Z(F_5) \cap F'_5$.
4. $H_5 = \langle b^5 \rangle$.
5. $K_5 = \langle bab^{-1}a^3, ab^3a^2b^{-1}a^2b^{-1}, a^{-2}b^{-3}a^4b^{-1}ab^{-2}a^4b^{-1}a^4b^{-1}a^{-2}ba^{-5}b^{-3}a^{-1} \rangle$.
6. $L_5 = \langle a \rangle$.
7. In the CAYLEY program 1 card(s) and card(t) must be greater than 6.

CHAPTER 5

DIFFERENT EFFICIENT PRESENTATIONS FOR CERTAIN OF THE GROUPS SL(2,p)

This chapter will consist of two sections. In the first section different efficient presentations will be given for SL(2,p), where p is an odd prime number and $p \in \{ 5,7,11,13,17,19,23,29,31,41,43,53,79,89,109,139,229 \}$.

In the second different efficient presentations will be given for SL(2,p), where p is a prime power and $p \in \{ 8,16,25,27,49,169 \}$.

5.1 DIFFERENT EFFICIENT PRESENTATIONS FOR SL(2,p) where p belongs to the set $\{ 5,7,11,13,17,19,23,29,31,41,43,53,79,89,109,139,229 \}$:

The problem of finding efficient presentations for covering groups of finite simple groups is, in general, harder than finding efficient presentations for the simple groups themselves. In 1972 Sunday in [36] gave two generator and three relation presentations for the groups SL(2,p), p prime. However this presentation is not efficient because SL(2,p) has a trivial Schur multiplier ([40] Theorem 25.5). In [16] by Coxeter SL(2,5) was shown to have deficiency zero presentation and SL(2,5) was the only finite non trivial perfect group known to have deficiency zero [23]. Later in 1980 C.M.Campbell and E.F.Robertson in [6] gave an efficient presentation for the groups SL(2,p), p odd prime.

In this section we study groups of the following types

(i) $\langle a,b \mid a^{-k}(ab)^2 = 1 , b^{m+t}a^{-n}b^m a^{-n} = a^r \rangle$ r is dependent on m and n

(ii) $\langle a,b \mid a^{-k}(ab)^2 = 1 , b^{m+t}a^{-n}(b^m a^{-n})^2 = a^r \rangle$ r is dependent on m and n

(ii) $\langle a,b \mid a^{-k}(a^2b^{-1})^2 = 1 , b^{m+t}a^{-n}(b^m a^{-n})^3 = a^r \rangle$ r is dependent on m and n

(iv) $\langle a,b \mid a^{-k}(a^2b^{-2})^2 = 1 , b^{m+t}a^{n}(b^m a^{n})^2 = a^r \rangle$ r is dependent on m and n

In this section we are going to show that for certain values of k,t,m,n and r these groups are efficient and isomorphic to the groups SL(2,p).

Theorem 5.1.1 **(i)** If ($m \equiv 2(\mathrm{mod}5)$ and $n \equiv 2(\mathrm{mod}5)$) then

$$G_1 = \langle a,b \mid a^{-5}(ab)^2 = 1 , b^{m+5}a^{-n}b^m a^{-n} = a^{(3m-2n+8)} \rangle \cong SL(2,5)$$

and this is an efficient presentation for G_1.

(ii) If ($m \equiv 3(\mathrm{mod}5)$ and $n \equiv 3(\mathrm{mod}5)$) then

$$G_1 = \langle a,b \mid a^{-5}(ab)^2 = 1 , b^{m+5}a^{-n}b^m a^{-n} = a^{(3m-2n+7)} \rangle \cong SL(2,5)$$

and this is an efficient presentation for G_1.

Proof: (i) $m \equiv 2(\mathrm{mod}5)$ and $n \equiv 2(\mathrm{mod}5)$:

If $m \equiv 2(\mathrm{mod}5)$ and $n \equiv 2(\mathrm{mod}5)$ then $G_1 = G'_1$ and consequently $a^5 \in G_1 = G'_1$. From relation 1 it can be easily seen that a^5 commutes with b, therefore $a^5 \in Z(G_1)$. Since $a^5 \in G'_1$ it can be seen that $a^5 \in Z(G_1) \cap G'_1$.

Now consider the homomorphic image of G_1 by $H_i = \langle a^5 \rangle$, in other words

$$G_1/\langle a^5 \rangle = \langle a,b \mid a^5 = (ab)^2 = b^{m+5}a^{-n}b^m a^{-n} = 1 \rangle.$$

By Theorem 4.1.1 $G_1/\langle a^5 \rangle$ is isomorphic to PSL(2,5).

On the other hand the Schur multiplier of PSL(2,5) is C_2 and since G_1 has two generators and two relations G_1 cannot be isomorphic to PSL(2,5) and so it has to be isomorphic to SL(2,5).

G_1 has two generators, two relations and its Schur multiplier is the trivial group. Therefore this presentation for G_1 is efficient.

Proof: (ii) $m \equiv 3(\text{mod}5)$ and $n \equiv 3(\text{mod}5)$:

If $m \equiv 3(\text{mod}5)$ and $n \equiv 3(\text{mod}5)$ then $G_1 = G'_1$ and consequently $a^5 \in G_1 = G'_1$. From relation 1 it can be easily seen that a^5 commutes with b and therefore $a^5 \in Z(G_1)$. Since $a^5 \in G'_1$ it can be seen that $a^5 \in Z(G_1) \cap G'_1$.

Now consider the homomorphic image of G_1 by $H_i = \langle a^5 \rangle$ in other words

$$G_1/\langle a^5 \rangle = \langle a,b \mid a^5 = (ab)^2 = b^{m+5}a^{-n}b^m a^{-n} = 1 \rangle.$$

By Theorem 4.1.1 $G_1/\langle a^5 \rangle$ is isomorphic to PSL(2,5).

On the other hand the Schur multiplier of PSL(2,5) is C_2 and since G_1 has two generators and two relations G_1 cannot be isomorphic to PSL(2,5) and so it has to be isomorphic to SL(2,5).

G_1 has two generators, two relations and its Schur multiplier is the trivial group. Therefore this presentation for G_1 is efficient.

In the following sixteen theorems full proofs are essentially the same as the proof of Theorem 5.1.1 with slight modifications. In every case instead of full proofs only the modifications will be given which have to be made in the proof of Theorem 5.1.1 in order to obtain the full proof. These modifications are

1. The conditions which make G_i perfect.
2. The proof that a^r commutes with b.
3. The element a^r which is in $Z(G_i) \cap G'_i$.
4. The subgroup H_i which is going to be used to construct the homomorphic

image $G_i/\langle a^r \rangle$.

5. The related Theorem 4.1.i which is going to be used in showing, that $G_i/\langle a^r \rangle \cong PSL(2,p)$.

Theorem 5.1.2 If $m \equiv 1(\bmod 21)$ and $n \equiv 1(\bmod 7)$ then

$$G_2 = \langle a,b \mid a^7(a^2b^{-1})^{-2} = 1 ,\ b^{m+3}a^{-n}(b^m a^{-n})^3 = a^{(-6m-2n-4)} \rangle \cong SL(2,7)$$

and this is an efficient presentation for G_2.

Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m \equiv 1(\bmod 21)$ and $n \equiv 1(\bmod 7)$ then $G_2 = G'_2$.
2. From relation 1 it can be easily seen that a^7 commutes with b .
3. If $m \equiv 1(\bmod 21)$ and $n \equiv 1(\bmod 7)$ then $a^7 \in Z(G_2) \cap G'_2$.
4. $H_2 = \langle a^7 \rangle$.
5. $G_2/\langle a^7 \rangle \cong PSL(2,7)$, by Theorem 4.1.2 .

Theorem 5.1.3 If $m = 10t + 2$ and $n \equiv 2(\bmod 5)$ then

$$G_3 = \langle a,b \mid a^5(ab)^{-2} = 1 ,\ b^{m+5}a^{-n}(b^m a^{-n})^2 = a^{((9/2)m-3n+7)} \rangle \cong SL(2,11)$$

and this is an efficient presentation for G_3.

Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m = 10t + 2$ and $n \equiv 2(\bmod 5)$ then $G_3 = G'_3$.
2. From relation 1 it can be easily seen that a^5 commutes with b .
3. If $m = 10t + 2$ and $n \equiv 2(\bmod 5)$ then $a^5 \in Z(G_3) \cap G'_3$.
4. $H_3 = \langle a^7 \rangle$.
5. $G_3/\langle a^5 \rangle \cong PSL(2,11)$, by Theorem 4.1.3 .

Theorem 5.1.4 If $m \equiv 51(\mathrm{mod} 91)$ and $n \equiv 7(\mathrm{mod} 13)$ then

$$G_4 = \langle a,b \mid a^{13}(ab)^{-2} = 1 , b^{m+7}a^{-n}b^{m}a^{-n} = a^{(11m-2n+38)} \rangle \cong SL(2,13)$$

and this is an efficient presentation for G_4.

Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m \equiv 51(\mathrm{mod} 91)$ and $n \equiv 7(\mathrm{mod} 13)$ then $G_4 = G'_4$.
2. From relation 1 it can be easily seen that a^{13} commutes with b .
3. If $m \equiv 51(\mathrm{mod} 91)$ and $n \equiv 7(\mathrm{mod} 13)$ then $a^{13} \in Z(G_4) \cap G'_4$.
4. $H_4 = \langle a^{13} \rangle$.
5. $G_4/\langle a^{13} \rangle \cong PSL(2,13)$, by Theorem 4.1.4 .

Theorem 5.1.5 **(i)** If $m \equiv 7(\mathrm{mod} 27)$ and $n \equiv 1(\mathrm{mod} 9)$ then

$$G_5 = \langle a,b \mid a^{9}(a^2b^{-1})^{-2} = 1 , b^{m+3}a^{-n}(b^{m}a^{-n})^{3} = a^{(-10m-4n-7)} \rangle \cong SL(2,17)$$

and this is an efficient presentation for G_5.

(ii) If $m \equiv 5(\mathrm{mod} 27)$ and $n \equiv 8(\mathrm{mod} 9)$ then

$$G_5 = \langle a,b \mid a^{9}(a^2b^{-1})^{-2} = 1 , b^{m+3}a^{-n}(b^{m}a^{-n})^{3} = a^{(26m-4n+19)} \rangle \cong SL(2,17)$$

and this is an efficient presentation for G_5.

Case (i): Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m \equiv 7(\mathrm{mod} 27)$ and $n \equiv 1(\mathrm{mod} 9)$ $G_5 = G'_5$.
2. From relation 1 it can be easily seen that a^9 commutes with b .
3. If $m \equiv 7(\mathrm{mod} 27)$ and $n \equiv 1(\mathrm{mod} 9)$ then $a^9 \in Z(G_5) \cap G'_5$.
4. $H_5 = \langle a^9 \rangle$.
5. $G_5/\langle a^9 \rangle \cong PSL(2,17)$, by Theorem 4.1.5 .

Case (ii): Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m \equiv 5(\mathrm{mod} 27)$ and $n \equiv 8(\mathrm{mod} 9)$ $G_5 = G'_5$.
2. From relation 1 it can be easily seen that a^9 commutes with b .

3. If $m \equiv 5 \pmod{27}$ and $n \equiv 8 \pmod{9}$ then $a^9 \in Z(G_5) \cap G'_5$.

4. $H_5 = \langle a^9 \rangle$.

5. $G_5/\langle a^9 \rangle \cong PSL(2,17)$, by Theorem 4.1.5 .

Theorem 5.1.6 If $m \equiv 7 \pmod{45}$ and $n \equiv 2 \pmod{5}$ then

$$G_6 = \langle\, a,b \mid a^5(ab)^{-2} = 1 \,,\, b^{m+9}a^{-n}b^m a^{-n} = a^{(3m-2n+13)} \,\rangle \cong SL(2,19)$$

and this is an efficient presentation for G_6.

Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m \equiv 7 \pmod{45}$ and $n \equiv 2 \pmod{5}$ then $G_6 = G'_6$.

2. From relation 1 it can be easily seen that a^5 commutes with b .

3. If $m \equiv 7 \pmod{45}$ and $n \equiv 2 \pmod{5}$ then $a^5 \in Z(G_6) \cap G'_6$.

4. $H_6 = \langle a^5 \rangle$.

5. $G_6/\langle a^5 \rangle \cong PSL(2,19)$, by Theorem 4.1.6 .

Theorem 5.1.7 If $m \equiv 29 \pmod{33}$ and $n \equiv 3 \pmod{11}$ then

$$G_7 = \langle\, a,b \mid a^{-11}(a^2b^{-1})^2 = 1 \,,\, b^{m+3}a^{-n}(b^m a^{-n})^3 = a^{(-14m-4n-11)} \rangle \cong SL(2,23)$$

and this is an efficient presentation for G_7 .

Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m \equiv 29 \pmod{33}$ and $n \equiv 3 \pmod{11}$ then $G_7 = G'_7$.

2. From relation 1 it can be easily seen that a^{11} commutes with b .

3. If $m \equiv 29 \pmod{33}$ and $n \equiv 3 \pmod{11}$ then $a^{11} \in Z(G_7) \cap G'_7$.

4. $H_7 = \langle a^{11} \rangle$.

5. $G_7/\langle a^{11} \rangle \cong PSL(2,23)$, by Theorem 4.1.7 .

Theorem 5.1.8 If $m \equiv 83(\mathrm{mod}105)$ and $n \equiv 2(\mathrm{mod}7)$ then

$$G_8 = \langle\, a,b \mid a^7(ab)^{-2} = 1\ ,\ b^{m+15}a^{-n}b^m a^{-n} \;=\; a^{(5m-2n+37)} \,\rangle \cong SL(2,29)$$

and this is an efficient presentation for G_8.

Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m \equiv 83(\mathrm{mod}105)$ and $n \equiv 2(\mathrm{mod}7)$ then $G_8 = G'_8$.
2. From relation 1 it can be easily seen that a^7 commutes with b.
3. If $m \equiv 83(\mathrm{mod}105)$ and $n \equiv 2(\mathrm{mod}7)$ then $a^7 \in Z(G_8) \cap G'_8$.
4. $H_8 = \langle a^7 \rangle$.
5. $G_8/\langle a^7 \rangle \cong PSL(2,29)$, by Theorem 4.1.8 .

Theorem 5.1.9 If $m = 40t + 1$, $t \in \mathbb{Z}$ and $n \equiv 1(\mathrm{mod}5)$ then

$$G_9 = \langle\, a,b \mid a^{-5}(a^2b^{-2})^2 = 1\ ,\ b^{m+8}a^n b^m a^n b^m a^n \;=\; a^{(3n-30t-3)} \,\rangle \cong SL(2,31)$$

and this is an efficient presentation for G_9.

Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m = 40t + 1$, $t \in \mathbb{Z}$ and $n \equiv 1(\mathrm{mod}5)$ then $G_9 = G'_9$.
2. From relation 1 it can be easily seen that a^5 commutes with b^2 and using relation 2 it can be seen that a^5 commutes with b.
3. If $m = 40t + 1$, $t \in \mathbb{Z}$ and $n \equiv 1(\mathrm{mod}5)$ then $a^5 \in Z(G_9) \cap G'_9$.
4. $H_9 = \langle a^5 \rangle$.
5. $G_9/\langle a^5 \rangle \cong PSL(2,31)$, by Theorem 4.1.9 .

Theorem 5.1.10 If $m \equiv 16(\mathrm{mod}105)$ and $n \equiv 2(\mathrm{mod}5)$ then

$$G_{10} = \langle\, a,b \mid a^5(ab)^{-2} = 1\ ,\ b^{m+21}a^{-n}b^m a^{-n} \;=\; a^{(3m-2n+31)} \,\rangle \cong SL(2,41)$$

and this is an efficient presentation for G_{10}.

Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m \equiv 16(\mathrm{mod}105)$ and $n \equiv 2(\mathrm{mod}5)$ then $G_{10} = G'_{10}$.
2. From relation 1 it can be easily seen that a^5 commutes with b.
3. If $m \equiv 16(\mathrm{mod}105)$ and $n \equiv 2(\mathrm{mod}5)$ then $a^5 \in Z(G_{10}) \cap G'_{10}$.
4. $H_{10} = \langle a^5 \rangle$.
5. $G_{10}/\langle a^5 \rangle \cong PSL(2,41)$, by Theorem 4.1.11 .

Theorem 5.1.11 If $m \equiv 57(\mathrm{mod}77)$ and $n \equiv 2(\mathrm{mod}7)$ then

$$G_{11} = \langle a,b \mid a^7(ab)^{-2} = 1 ,\ b^{m+11}a^{-n}b^m a^{-n} = a^{(5m-2n+27)} \rangle \cong SL(2,43)$$

and this is an efficient presentation for G_{11}.

Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m \equiv 57(\mathrm{mod}77)$ and $n \equiv 2(\mathrm{mod}7)$ then $G_{11} = G'_{11}$.
2. From relation 1 it can be easily seen that a^7 commutes with b.
3. If $m \equiv 57(\mathrm{mod}77)$ and $n \equiv 2(\mathrm{mod}7)$ then $a^7 \in Z(G_{11}) \cap G'_{11}$.
4. $H_{11} = \langle a^7 \rangle$.
5. $G_{11}/\langle a^7 \rangle \cong PSL(2,43)$, by Theorem 4.1.12 .

Theorem 5.1.12 If $m \equiv 12(\mathrm{mod}117)$ and $n \equiv 6(\mathrm{mod}13)$ then

$$G_{12} = \langle a,b \mid a^{13}(ab)^{-2} = 1 ,\ b^{m+9}a^{-n}b^m a^{-n} = a^{(11m-2n+49)} \rangle \cong SL(2,53)$$

and this is an efficient presentation for G_{12}.

Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m \equiv 12(\mathrm{mod}117)$ and $n \equiv 6(\mathrm{mod}13)$ then $G_{12} = G'_{12}$.
2. From relation 1 it can be easily seen that a^{13} commutes with b.
3. If $m \equiv 12(\mathrm{mod}117)$ and $n \equiv 6(\mathrm{mod}13)$ then $a^{13} \in Z(G_{12}) \cap G'_{12}$.
4. $H_{12} = \langle a^{13} \rangle$.

5. $G_{12}/\langle a^{13} \rangle \cong PSL(2,53)$, by Theorem 4.1.13 .

Theorem 5.1.13 If $m \equiv 28 \pmod{39}$ and $n \equiv 7 \pmod{13}$ then

$$G_{13} = \langle a,b \mid a^{-13}(a^2b^{-1})^2 = 1 ,\ b^{m+3}a^{-n}(b^ma^{-n})^3 = a^{(-18m-4n-14)} \rangle \cong SL(2,79)$$

and this is an efficient presentation for G_{13} .

Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m \equiv 28 \pmod{39}$ and $n \equiv 7 \pmod{13}$ then $G_7 = G'_7$.
2. From relation 1 it can be easily seen that a^{13} commutes with b .
3. If $m \equiv 28 \pmod{39}$ and $n \equiv 7 \pmod{13}$ then $a^{13} \in Z(G_{13}) \cap G'_{13}$.
4. $H_{13} = \langle a^{13} \rangle$.
5. $G_{13}/\langle a^{13} \rangle \cong PSL(2,79)$, by Theorem 4.1.15 .

Theorem 5.1.14 If $m \equiv 48 \pmod{99}$ and $n \equiv 5 \pmod{11}$ then

$$G_{14} = \langle a,b \mid a^{11}(ab)^{-2} = 1 ,\ b^{m+9}a^{-n}b^ma^{-n} = a^{(9m-2n+40)} \rangle \cong SL(2,89)$$

and this is an efficient presentation for G_{14}.

Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m \equiv 48 \pmod{99}$ and $n \equiv 5 \pmod{11}$ then $G_{14} = G'_{14}$.
2. From relation 1 it can be easily seen that a^{11} commutes with b .
3. If $m \equiv 48 \pmod{99}$ and $n \equiv 5 \pmod{11}$ then $a^{11} \in Z(G_{14}) \cap G'_{14}$.
4. $H_{14} = \langle a^{11} \rangle$.
5. $G_{14}/\langle a^{11} \rangle \cong PSL(2,89)$, by Theorem 4.1.17 .

Theorem 5.1.15 If $m \equiv 35 \pmod{99}$ and $n \equiv 2 \pmod{9}$ then

$$G_{15} = \langle a,b \mid a^9(ab)^{-2} = 1 ,\ b^{m+11}a^{-n}b^ma^{-n} = a^{(7m-2n+38)} \rangle \cong SL(2,109)$$

and this is an efficient presentation for G_{15}.

Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m \equiv 35(\mathrm{mod}99)$ and $n \equiv 2(\mathrm{mod}9)$ then $G_{15} = G'_{15}$.
2. From relation 1 it can be easily seen that a^9 commutes with b .
3. If $m \equiv 35(\mathrm{mod}99)$ and $n \equiv 2(\mathrm{mod}9)$ then $a^9 \in Z(G_{15}) \cap G'_{15}$.
4. $H_{15} = \langle a^9 \rangle$.
5. $G_{15}/\langle a^9 \rangle \cong PSL(2,109)$, by Theorem 4.1.18 .

Theorem 5.1.16 If $m \equiv 30(\mathrm{mod}115)$ and $n \equiv 2(\mathrm{mod}5)$ then

$$G_{16} = \langle a,b \mid a^5(ab)^{-2} = 1\ ,\ b^{m+23}a^{-n}b^m a^{-n} = a^{(3m-2n+34)} \rangle \cong SL(2,139)$$

and this is an efficient presentation for G_{16}.

Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m \equiv 30(\mathrm{mod}115)$ and $n \equiv 2(\mathrm{mod}5)$ then $G_{16} = G'_{16}$.
2. From relation 1 it can be easily seen that a^5 commutes with b .
3. If $m \equiv 30(\mathrm{mod}115)$ and $n \equiv 2(\mathrm{mod}5)$ then $a^5 \in Z(G_{16}) \cap G'_{16}$.
4. $H_{16} = \langle a^5 \rangle$.
5. $G_{16}/\langle a^5 \rangle \cong PSL(2,139)$, by Theorem 4.1.19 .

Theorem 5.1.17 If $m \equiv 52(\mathrm{mod}95)$ and $n \equiv 7(\mathrm{mod}19)$ then

$$G_{17} = \langle a,b \mid a^{19}(ab)^{-2} = 1\ ,\ b^{m+5}a^{-n}b^m a^{-n} = a^{(17m-2n+42)} \rangle \cong SL(2,229)$$

and this is an efficient presentation for G_{17}.

Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m \equiv 52(\mathrm{mod}95)$ and $n \equiv 7(\mathrm{mod}19)$ then $G_{17} = G'_{17}$.
2. From relation 1 it can be easily seen that a^{19} commutes with b .
3. If $m \equiv 52(\mathrm{mod}95)$ and $n \equiv 7(\mathrm{mod}19)$ then $a^{19} \in Z(G_{17}) \cap G'_{17}$.
4. $H_{17} = \langle a^{19} \rangle$
5. $G_{17}/\langle a^{19} \rangle \cong PSL(2,229)$, by Theorem 4.1.20 .

5.2 DIFFERENT EFFICIENT PRESENTATIONS FOR THE GROUPS $SL(2,p^n)$ where $p^n \in \{ 8,16,25,27,49,169 \}$:

In general the efficiency of $SL(2,p^n)$ hasn't been solved yet. In the literature we can find nothing, except for a few cases of particular values of p and n. In [6] C.M.Campbell and E.F.Robertson have given efficient presentation for SL(2,8) and in [8] by the same authors the efficiency of SL(2,16) is shown. The efficiency of SL(2,25), SL(2,27), SL(2,32), SL(2,49) and SL(2,64) is also shown in [9]. In 1988 in [11] the efficiency of SL(2,169) was given by C.M.Campbell and E.F.Robertson.

In this section we study groups of the following types

(i) $\langle a,b \mid a^{-k}(ab)^2 = 1,\ b^{63m+11}a^{-n}b^{63m+2}a^{-n} = a^r \rangle$ r is dependent on m and n

(ii) $\langle a,b \mid a^{-k}(ab)^2 = 1,\ b^{m+t}a^{-n}b^{m}a^{-n} = a^r \rangle$ r is dependent on m and n.

In this section we are going to show that for certain values of k,t,m,n and r these groups are efficient and isomorphic to the groups $SL(2,p^n)$.

Theorem 5.2.1 If $m \in \mathbb{Z}$ and $n \equiv 2 \pmod 7$ then

$G_1 = \langle a,b \mid a^7(ab)^{-2} = 1,\ b^{(63m+11)}a^{-n}b^{(63m+2)}a^{-n} = a^{(315m-2n+32)} \rangle \cong SL(2,8)$

and this is an efficient presentation for G_1.

Proof: If $m \in \mathbb{Z}$ and $n \equiv 2 \pmod 7$ then $G_1 = G'_1$ and consequently a^7, $b^9 \in G_1 = G'_1$. From relation 1 it can be seen that a^7 commutes with b. Using this information with relation 2 it can be seen that b^9 commutes with a^2. Again using relation 1

$$a^7 = (ab)^2$$

$$a^6 = bab$$

$$b^9a^6 = b^9bab \ldots\ldots\ldots\ldots (*)$$

$a^6b^9 = babb^9$............ (**).

From (*) and (**) it can be seen that b^9 commutes with a. Since a^7, $b^9 \in G_1 = G'_1$ then a^7, $b^9 \in Z(G_1) \cap G'_1$.

Now consider the factor group

$$G_1/\langle a^7,b^9 \rangle \cong \langle a,b \mid a^7 = (ab)^2 = (b^2a^{-2})^2 = b^9 = 1 \rangle$$

Using the CAYLEY program 1 it can be seen that $G_1/\langle a^7,b^9 \rangle$ is isomorphic to SL(2,8). In the CAYLEY program 1 the presentation for SL(2,8) has been taken from C.M.Campbell and E.F.Robertson [6]. Since $G_1/\langle a^7,b^9 \rangle \cong SL(2,8)$ therefore it can be deduced that $\langle a^7,b^9 \rangle \leq M(SL(2,8))$. But the Schur multiplier of SL(2,8) is trivial. This means $| \langle a^7,b^9 \rangle | = 1$. So a^7, b^9 must be equal to identity element of G_1 and consequently $G_1 \cong SL(2,8)$. Since the Schur multiplier of G_1 is trivial this yields the claimed result.

Theorem 5.2.2 If $m \equiv 4(\text{mod}15)$ and $n \equiv 2(\text{mod}5)$ then

$$G_2 = \langle a,b \mid a^5(ab)^{-2} = 1 , b^{m+15}a^{-n}b^ma^{-n} = a^{(3m-2n+22)} \rangle \cong SL(2,16)$$

and this is an efficient presentation for G_2.

Proof: If $m \equiv 4(\text{mod}15)$ and $n \equiv 2(\text{mod}5)$ then $G_2 = G'_2$ and consequently a^5, $b^{15} \in G_2 = G'_2$. From relation 1 it can be seen that a^5 commutes with b. Using this information with relation 2 it can be seen that b^{15} commutes with a^2. Again using relation 1

$a^5 = (ab)^2$

$a^4 = bab$

$b^{15}a^4 = b^{15}bab$............ (*)

$a^4b^{15} = babb^{15}$............ (**)

From (*) and (**) it can be seen that eventually b^{15} commutes with a. Since a^5, $b^{15} \in G_2 = G'_2$ therefore a^5, $b^{15} \in Z(G_2) \cap G'_2$.

Now consider the factor group

$$G_2/\langle a^5,b^{15}\rangle \cong \langle a,b \mid a^5 = (ab)^2 = (b^4a^{-2})^2 = b^{15} = 1\rangle$$

Using the CAYLEY program 1 it can be seen that $G_1/\langle a^5,b^{15}\rangle$ is isomorphic to SL(2,16). In the CAYLEY program 1 the presentation for SL(2,16) has been taken from C.M.Campbell and E.F.Robertson [6]. Since $G_2/\langle a^5,b^{15}\rangle \cong SL(2,16)$ therefore it can be deduced that $\langle a^5,b^{15}\rangle \leq M(SL(2,16))$. But the Schur multiplier of SL(2,16) is trivial. This means $|\langle a^5,b^{15}\rangle| = 1$. So a^5, b^{15} must be equal to the identity element of G_2 and consequently $G_2 \cong SL(2,16)$. Since the Schur multiplier of G_2 is trivial the result follows as claimed.

In the following four theorems full proofs are essentially the same as the proof of Theorem 5.1.1 with slight modifications. In every case instead of full proofs only the modifications will be given which have to be made in the proof of Theorem 5.1.1 in order to obtain the full proof. These modifications are

1. The conditions which makes G_i perfect.
2. The proof that a^r commutes with b.
3. The element a^r which is in $Z(G_i) \cap G_i'$.
4. The subgroup H_i which is going to be used to construct the homomorphic image $G_i/\langle a^r\rangle$.
5. The related Theorem 4.2.i which is going to be used in showing, that $G_i/\langle a^r\rangle \cong PSL(2,p)$.

Theorem 5.2.3 If $m \equiv 2 (\text{mod} 65)$ and $n \equiv 5 (\text{mod} 13)$ then

$$G_3 = \langle a,b \mid a^{13}(ab)^{-2} = 1 ,\ b^{m+5}a^{-n}b^m a^{-n} = a^{(11m-2n+27)}\rangle \cong SL(2,25)$$

and this is an efficient presentation for G_3.

Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m \equiv 2 \pmod{65}$ and $n \equiv 5 \pmod{13}$ then $G_3 = G'_3$.
2. From relation 1 it can be easily seen that a^{13} commutes with b.
3. If $m \equiv 2 \pmod{65}$ and $n \equiv 5 \pmod{13}$ then $a^{13} \in Z(G_3) \cap G'_3$.
4. $H_3 = \langle a^{13} \rangle$.
5. $G_3/\langle a^{13} \rangle \cong PSL(2,25)$, by Theorem 4.2.2 .

Theorem 5.2.4 If $m \equiv 55 \pmod{91}$ and $n \equiv 3 \pmod{7}$ then

$$G_4 = \langle\, a,b \mid a^7(ab)^{-2} = 1 \,,\; b^{m+13}a^{-n}b^{m}a^{-n} = a^{(5m-2n+32)} \,\rangle \cong SL(2,27)$$

and this is an efficient presentation for G_4.

Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m \equiv 55 \pmod{91}$ and $n \equiv 3 \pmod{7}$ then $G_4 = G'_4$.
2. From relation 1 it can be easily seen that a^7 commutes with b .
3. If $m \equiv 55 \pmod{91}$ and $n \equiv 3 \pmod{7}$ then $a^7 \in Z(G_4) \cap G'_4$.
4. $H_4 = \langle a^7 \rangle$.
5. $G_4/\langle a^7 \rangle \cong PSL(2,27)$, by Theorem 4.2.3.

Theorem 5.2.5 If $m \equiv 14 \pmod{25}$ and $n \equiv 2 \pmod{5}$ then

$$G_5 = \langle\, a,b \mid a^5(ab)^{-2} = 1 \,,\; b^{m+25}a^{-n}b^{m}a^{-n} = a^{(3m-2n+37)} \,\rangle \cong SL(2,49)$$

and this is an efficient presentation for G_5 .

Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m \equiv 14 \pmod{25}$ and $n \equiv 2 \pmod{5}$ then $G_5 = G'_5$.
2. From relation 1 it can be easily seen that a^5 commutes with b .
3. If $m \equiv 14 \pmod{25}$ and $n \equiv 2 \pmod{5}$ then $a^5 \in Z(G_5) \cap G'_5$.
4. $H_5 = \langle a^5 \rangle$.
5. $G_5/\langle a^5 \rangle \cong PSL(2,49)$, by Theorem 4.2.4 .

Theorem 5.2.6 If $m \equiv 82(\mathrm{mod}85)$ and $n \equiv 13(\mathrm{mod}17)$ then

$$G_6 = \langle a,b \mid a^{17}(ab)^{-2} = 1 \text{ , } b^{m+5}a^{-n}b^{m}a^{-n} = a^{(15m-2n+37)} \rangle \cong SL(2,169)$$

and this is an efficient presentation for G_5 .

Modifications which have to be made in Theorem 5.1.1 are as follows:

1. If $m \equiv 82(\mathrm{mod}85)$ and $n \equiv 13(\mathrm{mod}17)$ then $G_6 = G'_6$.
2. From relation 1 it can be easily seen that a^{17} commutes with b .
3. If $m \equiv 82(\mathrm{mod}85)$ and $n \equiv 13(\mathrm{mod}17)$ then $a^{17} \in Z(G_6) \cap G'_6$.
4. $H_6 = \langle a^{17} \rangle$.
5. $G_6/\langle a^{17} \rangle \cong PSL(2,169)$, by Theorem 4.2.5 .

CHAPTER 6

THE GROUP $\langle a,b \mid a^p = 1,\ b^{m+p}a^{-m}b^m a^{-m} = 1,\ (ab)^2 = 1 \rangle$

In this chapter we study the groups with the presentation of the form $\langle a,b \mid a^p = 1,\ b^{m+p}a^{-m}b^m a^{-m} = 1,\ (ab)^2 = 1 \rangle$, p an odd number and $m \in \mathbb{Z}$.

Lemma 6.1 Let $F = \langle a,b \mid a^p = 1,\ b^{m+k}a^{-m}b^m a^{-m} = 1,\ (ab)^2 = 1 \rangle$ and p is odd.

(i) If $(p,m) = 1$ then b^k commutes with a.

(ii) If $(p,m) = d$ then b^k commutes with a^d.

Proof (i): (i) is a special case of (ii).

(ii): From relation 2 $a^m b^k = b^k a^m$. If $(p,m) = d$ then b^k commutes with a^d.

Theorem 6.1 Let $F = \langle a,b \mid a^p = 1,\ b^{m+p}a^{-m}b^m a^{-m} = 1,\ (ab)^2 = 1 \rangle$ and p is odd. If $m \equiv 1(\mathrm{mod}\,p)$ or $m \equiv (p-1)(\mathrm{mod}\,p)$ then F is the trivial group.

Proof: Case $m \equiv 1(\mathrm{mod}\,p)$ Consider relation 2 and using relation 1 we can rewrite relation 2 as $b^{m+p}a^{-1}b^m a^{-1} = 1$. Since $(m,p) = 1$ by Lemma 6.1 b^p commutes with a.

Hence $b^{2m+p}a^{-1}ba^{-1} = 1$ using relation 3 as $a^{-1} = bab$

$\Rightarrow b^{2m+p}b^{-1}a^{-1}ba^{-1} = 1$

$\Rightarrow b^{2m+p}ab^2a^{-1} = 1$

$\Rightarrow ab^2a^{-1} = b^{-2}b^{kp}$ where b^{kp} is central

Raise both sides of the equation to the power (p+1)|2 to see that $aba^{-1} = b^t$,

so $\langle b \rangle \lhd F$. $F/\langle b \rangle$ is trivial. Hence F is a cyclic group and so its order is given by the invariant factors of the relation matrix

$$M = \begin{pmatrix} p & 0 \\ 2 & 2 \\ -2m & 2m+p \end{pmatrix}$$

1 is the only invariant factor of M

$h_1(M)$ has to be 1 because h.c.f ({ p, 0, 2, −2m, 2m+p }) = 1

$h_2(M)$ has to be 1 because h.c.f ({ 2p, $2mp + p^2$, 8m + 2p }) = 1. Hence F is the trivial group.

Case m ≡ (p−1)(modp): Claim: If m ≡ (p−1)(modp) then F is isomorphic to

$$\langle a,b \mid a^p = 1,\ b^{n+p}a^{-n}b^na^{-n} = 1,\ (ab)^2 = 1 \rangle$$

where n ≡ 1(modp).

Let m ≡(p−1)(modp). Then

$$F = \langle a,b \mid a^p = 1,\ b^{m+p}a^{-m}b^ma^{-m} = 1,\ (ab)^2 = 1 \rangle.$$

Using the map $b \mapsto b^{-1}$, $a \mapsto a^{-1}$ we can get

$$F = \langle a,b \mid a^p = 1,\ b^{-m-p}a^mb^{-m}a^m = 1,\ (ab)^2 = 1 \rangle.$$

Now let $-m = p + n$

Since m ≡ (p−1)(modp) ⇒ p|(m − (p−1)) ⇒ p|(− n + 1) ⇒ p|(n − 1) ⇒ n ≡ 1(modp). Replacing $-m = n + p$ in F yields the result as claimed.

Theorem 6.2 Let $F = \langle a,b \mid a^p = 1,\ b^{m+p}a^{-m}b^m a^{-m} = 1,\ (ab)^2 = 1 \rangle$, p odd and $(p,m) = d>1$. If the number $k = 2d(2m+p)(1/2 + 1/d + 1/(2m+p) - 1)$ is 0 or negative then the group F is infinite.

Proof: Add the relation $a^d = 1$. Then F has a homomorphic image H where

$$H = \langle a,b \mid a^d = 1,\ b^{2m+p} = 1,\ (ab)^2 = 1 \rangle$$

and this is the polyhedral group which is discussed in [17]. Such a group is finite if the number $k = 2d(2m+p)(1/2 + 1/d + 1/(2m+p) - 1)$ is positive from which the result follows.

In the following three theorems for each of the values $p = 5,7,9$ we have investigated all possibilities for $m \in \mathbb{Z}$.

Theorem 6.3 Let $F = \langle a,b \mid a^5 = 1,\ b^{m+5}a^{-m}b^m a^{-m} = 1,\ (ab)^2 = 1 \rangle$. Then :

(i) If $m \equiv 4 \pmod 5$ or $m \equiv 1 \pmod 5$ then F is a trivial group.

(ii) If $m \equiv 0 \pmod 5$ then F is an infinite group.

(iii) If $m \equiv 2 \pmod 5$ or $m \equiv 3 \pmod 5$ then F is PSL(2,5).

Proof (i) Let $p = 5$ in Theorem 6.1 and the result is immediate.

(ii) Let $p = 5$ in Theorem 6.2. Then since

$k = 10(2m+5)(1/2 + 1/5 + 1/(2m+5) - 1) = -(6m+5)$

the result is immediate.

(iii) This case is considered in Theorem 4.1.1.

Theorem 6.4 Let $F = \langle a,b \mid a^7 = 1,\ b^{m+7}a^{-m}b^m a^{-m} = 1,\ (ab)^2 = 1 \rangle$ then

(i) If $m \equiv 6 \pmod 7$ or $m \equiv 1 \pmod 7$ then F is a trivial group.

(ii) If $m \equiv 0 \pmod 7$ then F is an infinite group.

(iii) If $m \equiv 2 \pmod 7$ or $m \equiv 5 \pmod 5$ or $m \equiv 3 \pmod 7$ or $m \equiv 4 \pmod 5$ then F is PSL(2,13).

Proof (i) Let $p = 7$ in Theorem 6.1 and the result is immediate.

(ii) Let $p = 7$ in Theorem 6.2. Then since

$k = 14(2m+7)(1/2 + 1/7 + 1/(2m+7) - 1) = -(10m+11)$

the result is immediate.

(iii) Case $m \equiv 2 \pmod 7$:

In this case the full proof is essentially the same as the proof of Theorem 4.1.1 with slight modifications. Instead of the full proof only the modification will be given for Theorem 4.1.1 and for the CAYLEY program 1. These modifications are:

1. If $m \equiv 2 \pmod 7$ then $F = F'$.
2. From relation 2 it can be easily seen that b^7 commutes with a.
3. If $m \equiv 2 \pmod 7$ then $b^7 \in Z(F) \cap F'$.
4. $H = \langle b^7 \rangle$.
5. $K = \langle a^{-1}b^{-1}, b^{-3}a^{-1} \rangle$.
6. $L = \langle a \rangle$.
7. Generating pair for SL(2,13) is (a,ab).

In the CAYLEY program 1 card(s) and card(t) must be greater than 1.

Case $m \equiv 5 \pmod 7$:

Using the map $b \mapsto b^{-1}$, $a \mapsto a^{-1}$ it can be seen that F is isomorphic to

$$\langle a,b \mid a^7 = (ab)^2 = b^{n+7}a^{-n}b^n a^{-n} = 1 \rangle \cong \mathrm{PSL}(2,13)$$

where $n \equiv 2 \pmod 7$.

Case $m \equiv 3(\text{mod}7)$:

In this case the full proof is essentially the same as the proof of Theorem 4.1.1 with slight modifications. Instead of the full proof only the modifications will be given for Theorem 4.1.1 and for the CAYLEY program 1. These modifications are:

1. If $m \equiv 3(\text{mod}7)$ then $F = F'$.
2. From relation 2 it can be easily seen that b^7 commutes with a.
3. If $m \equiv 3(\text{mod}7)$ then $b^7 \in Z(F) \cap F'$.
4. $H = \langle b^7 \rangle$.
5. $K = \langle a^{-1}b^{-1}, b^{-4}a^{-2} \rangle$.
6. $L = \langle a \rangle$.
7. Generating pair for SL(2,13) is (a,ab).

In the CAYLEY program 1 card(s) and card(t) must be greater than 1.

Case $m \equiv 4(\text{mod}7)$:

Using the map $b \mapsto b^{-1}$, $a \mapsto a^{-1}$ it can be seen that F is isomorphic to

$$\langle a,b \mid a^7 = (ab)^2 = b^{n+7}a^{-n}b^na^{-n} = 1 \rangle \cong PSL(2,13)$$

where $n \equiv 3(\text{mod}7)$.

Theorem 6.5 Let $F = \langle a,b \mid a^9 = 1,\ b^{m+9}a^{-m}b^ma^{-m} = 1,\ (ab)^2 = 1 \rangle$. Then:

(i) If $m \equiv 8(\text{mod}9)$ or $m \equiv 1(\text{mod}9)$ then F is a trivial group.

(ii) If $m \equiv 0(\text{mod}9)$ or $m \equiv 3(\text{mod}9)$ or $m \equiv 6(\text{mod}9)$ then F is an infinite group.

(iii) If $m \equiv 2(\text{mod}9)$ or $m \equiv 7(\text{mod}9)$ or $m \equiv 5(\text{mod}9)$ or $m \equiv 4(\text{mod}9)$ then F is PSL(2,37).

Proof (i) Let $p = 9$ in Theorem 6.1 and the result is immediate.

(ii) Case $m \equiv 0(\text{mod}9)$:

Let $p = 9$ in Theorem 6.2 and since

$k = 18(2m+9)(1/2 + 1/9 + 1/(2m + 9) - 1) = -(14m+63)$

the result is immediate.

Case $m \equiv 3 \pmod 9$:

Let $p = 9$ in Theorem 6.2 and since

$k = 6(2m+9)(1/2 + 1/3 + 1/(2m + 9) - 1) = -(2m+3)$

the result is immediate.

Case $m \equiv 6 \pmod 9$:

Let $p = 9$ in Theorem 6.2 and since

$k = 12(2m+9)(1/2 + 1/6 + 1/(2m + 9) - 1) = -(8m+24)$

the result is immediate.

(iii) This case is considered in Theorem 4.1.10.

CHAPTER 7

THE DIRECT PRODUCTS

This chapter will consist of three sections. In the first section for some values of n and m we shall show that the direct products $PSL(2,\mathbb{Z}_n) \times PSL(2,\mathbb{Z}_m)$ are efficient. In section two, for some values of p, we shall show that the direct products $PSL(2,\mathbb{Z}_p) \times PSL(2,3^2)$ are efficient. In the last section of this chapter we shall show the efficiency of the following direct products

(i) $PSL(2,5) \times PSL(2,3^2)$

(ii) $PSL(2,7) \times PSL(2,3^2)$

(iii) $PSL(2,5) \times PSL(2,3^3)$.

Additionally in the last section of this chapter we shall investigate the structure of the perfect group $\langle\, a,b \mid a^5 = 1,\ b^7 = 1,\ (b^2a^{-2})^2 = 1,\ (ab)^3 = 1 \,\rangle$ of order 161280.

7.1 INTRODUCTION:

Questions concerning the efficiency of direct products have been of considerable interest for a number of years. The first questions concerning the efficiency of direct products were posed by Wiegold in [41]. In particular his questions were whether $PSL(2,5) \times PSL(2,5)$ and $SL(2,5) \times SL(2,5)$ are efficient. The first of these questions was answered by Kenne in [26]. He showed that

$PSL(2,5) \times PSL(2,5)$ =

$$\langle a,b \mid a^{10} = b^6 = a^4ba^{-1}b^{-3}a^{-1}b^{-1} = (ab^2)^2a^{-1}b^{-1}(ab^{-1})^2 = 1 \rangle$$

is efficient .The second question was answered by Campbell et al. [9].

In general for every prime p the efficiency of $PSL(2,p) \times PSL(2,p)$ has been shown in [13] by C.M.Campbell, E.F.Robertson and P.D.Williams.

In [12] C.M.Campbell, E.F.Robertson and P.D.Williams have obtained efficient presentations for certain direct products involving fields of the same characteristic i.e. direct products of groups $PSL(2,p^{n_i})$ for a fixed prime p and different n_i's. Also in the same paper they have obtained efficient presentations for direct products involving fields of different characteristics i.e. efficient presentations for groups of the form $PSL(2,p_1) \times PSL(2,p_2)$, p_1,p_2 prime powers.

The problem of the efficiency of $G = PSL(2,p_1) \times PSL(2,p_2) \times \cdots \times PSL(2,p_n)$ where the p_i are distinct primes was solved. One can start from the efficient presentation for $SL(2,\mathbb{Z}_m)$, $m = p_1p_2...p_n$, given in [6], adding n relations providing the i^{th} relation effectively making the central element of order 2 in $SL(2,p_i)$ equal to 1.

In this chapter it is also worth noting the following comment for notation. For commutative ring R with a 1 define SL(2,R) to be the group of 2×2 matrices with determinant 1 over R. Define $PSL(2,R) = SL(2,R)/\{\pm I\}$ where I is the 2×2 identity matrix. If R is the finite field $GF(p^n)$, where p is a prime , we write $PSL(2,R) = PSL(2,p^n)$. The order of $PSL(2,p^n)$ is

$$p^n(p^n - 1)(p^n + 1)/2.$$

If R is the ring of integers modulo m, then we write $PSL(2,R) = PSL(2, \mathbb{Z}_m)$. In terms of the prime factorization $m = \prod p^c$, the order of $PSL(2, \mathbb{Z}_m)$ is, see [36],

$$m^3\prod p^c(1 - 1/p^2)/2.$$

7.2 THE DIRECT PRODUCT $PSL(2,\mathbb{Z}_n) \times PSL(2,\mathbb{Z}_m)$:

In this section we shall investigate the direct products $PSL(2,\mathbb{Z}_n) \times PSL(2,\mathbb{Z}_m)$, n,m odd numbers and

(i) $n \equiv 1(\mathrm{mod}6)$ and $m \equiv 1(\mathrm{mod}6)$

(ii) $n \equiv -1(\mathrm{mod}6)$ and $m \equiv -1(\mathrm{mod}6)$

(iii) $n \equiv 1(\mathrm{mod}6)$ and $m \equiv -1(\mathrm{mod}6)$

(iv) $n \equiv -1(\mathrm{mod}6)$ and $m \equiv 1(\mathrm{mod}6)$

in an attempt to prove that these groups are efficient. For a particular n,m we shall give efficient presentations which were not previously known to be efficient.

We consider in details case (iii). The other cases can be deduced from case (iii).

Let $G = PSL(2,\mathbb{Z}_n) \times PSL(2,\mathbb{Z}_m)$. Then, using a presentation for $PSL(2,\mathbb{Z}_p)$ given in [36].

$$\begin{aligned} G = \langle\ a,b,c,d \mid\ & a^2 = b^n = (ab)^3 = (ab^4ab^{(n+1)/2})^2 = 1, \\ & c^2 = d^m = (cd)^3 = (cd^4cd^{(m+1)/2})^2 = 1, \\ & [a,c] = [a,d] = [b,c] = [b,d] = 1 \quad \rangle \end{aligned}$$

Put $x = bcd$, $y = abd$. Then let $n = 6k + 1$, $m = 6t - 1$. We have $x^3 = b^3 \Rightarrow x^{n-1} = b^{-1}$ so $b = x^{1-n}$. Similarly $y^3 = d^3 \Rightarrow y^{m+1} = d^{m+1} = d$ so $d = y^{m+1}$. Since $x = bcd$ $\Rightarrow c = x^ny^{-m-1}$. Also since $y = abd \Rightarrow a = y^{-m}x^{n-1}$. We have proved:

Lemma 7.2.1 If $x = bcd$, $y = abd$ then $a = y^{-m}x^{n-1}$, $b = x^{1-n}$, $c = x^ny^{-m-1}$, $d = y^{m+1}$.

We write down the 12 relations of G written in terms of x and y in the order they appear in the presentation above

(1) $(y^{-m}x^{n-1})^2 = 1$

(2) $(x^{1-n})^n = 1$

(3) $y^{-3m} = 1$

(4) $(y^{-m}x^{-3n+3}y^{-m}x^{r})^2 = 1$, where $r = -(n-1)^2/2$

(5) $(x^n y^{-m-1})^2 = 1$

(6) $(y^{m+1})^m = 1$

(7) $x^{3n} = 1$

(8) $(x^n y^{3m+3} x^n y^s)^2 = 1$,where $s = (m^2 - 1)/2$

(9) $[y^{-m}x^{n-1}, x^n y^{-m-1}] = 1$

(10) $[y^{-m}x^{n-1}, y^{m+1}] = 1$

(11) $[x^{1-n}, x^n y^{-m-1}] = 1$

(12) $[x^{1-n}, y^{m+1}] = 1$

Lemma 7.2.2 In Lemma 7.2.1 the relations (2), (6), (10), (11) are redundant.

Proof: Since $3|(1-n)$ then x^{1-n} is a power of x^3 so, since $x^{3n} = 1$ we have $(x^{1-n})^2 = 1$. Since $3|(m+1)$ then y^{m+1} is a power of y^3 so, since $y^{3m} = 1$ we have $(y^{m+1})^m = 1$. Also (10) and (11) are immediate consequences of (12).

We now tidy up a little. Since $[x^{1-n}, y^{m+1}] = 1$, cubing these two elements we get $[x^3, y^3] = 1$. Also using (7) we can replace r in (4) by $(n-1)/2$. Using (3) we can replace s in (8) by $(-m-1)/2$. We now have the presentation for G as follows.

Lemma 7.2.3 G is generated by x and y subject to the relations

(i) $(y^{-m}x^{n-1})^2 = 1$

(ii) $(x^n y^{-m-1})^2 = 1$

(iii) $(y^{-m}x^3 y^{-m} x^{(n-1)/2})^2 = 1$

(iv) $(x^n y^3 x^n y^{-(m+1)/2})^2 = 1$

(v) $[x^n, y^m] = 1$

(vi) $[x^3, y^3] = 1$

(vii) $x^{3n} = 1$

(viii) $y^{3m} = 1$

Consider (i). We have $y^{-m}x^{n-1}y^{-m}x^{n-1} = 1$ and using (v) and (vii) this gives $y^{-m}x^{-1}y^{-m}x^{-n-1} = 1$. Hence $x^{-n-1} = y^m x y^m$ replaces (i). Similarly consider (ii). We have $x^n y^{-m-1} x^n y^{-m-1} = 1$ and using (v) and (viii) this gives $x^n y^{-1} x^n y^{m-1} = 1$. Hence $y^{-m+1} = x^n y^{-1} x^n$ replaces (ii). But consider again $y^{-m}x^{n-1}y^{-m}x^{n-1} = 1$ and this time use (vi) in the form $[x^{n-1}, y^{m+1}] = 1$. We have

$$y^{-m}x^{n-1}y^{-m-1}yx^{n-1} = 1 \Rightarrow y^{-2m-1}x^{n-1}yx^{n-1} = 1$$

$\Rightarrow y^{m-1}x^{n-1}yx^{n-1} = 1$ and substituting $y^{m-1} = x^{1-n}y^{-1}x^{1-n}$ into this relation gives $x^n y^{-1} x^n y^{m-1} = x^n y^{-1} x^n x^{1-n} y^{-1} x^{1-n} = xy^{-1}xy^{-1} = 1$ so $(xy^{-1})^2 = 1$. This now replaces $y^{-m+1} = x^n y^{-1} x^n$. Use this to replace

$x^{-n-1} = y^m x y^m$ by $x^{-n-1} = y^{m+1}x^{-1}y^{m+1}$. We have new relations (i)* and (ii)* to replace respectively (i) and (ii). They are

(i)* $x^{-n-1} = y^{m+1}x^{-1}y^{m+1}$

(ii)* $(xy^{-1})^2 = 1$.

Lemma 7.2.4 In Lemma 7.2.3 $[x^n, y^m] = 1$ and $[x^3, y^3] = 1$ are redundant.

Proof: Using the (ii)* we can rewrite (i)* as $x^{-n} = y^m x y^m x$ so

$[y^m x, x^n] = 1 \Rightarrow [y^m, x^n] = 1$.

Consider (i)* i.e. $x^{-n-1} = y^{m+1}x^{-1}y^{m+1} \Rightarrow x^{-n-2} = (y^{m+1}x^{-1})^2 \Rightarrow [y^{m+1}, x^{n+2}] = 1$.

Cubing the first term in $[y^{m+1}, x^{n+2}] = 1$ and using (viii) we have $[y^3, x^{n+2}] = 1$. Cubing the second term in $[y^3, x^{n+2}] = 1$ and using (vii) we have $[y^3, x^6] = 1$. Now considering $[y^3, x^{n-1+3}] = 1$ and using the fact that $6|(n-1)$ and using $[y^3, x^6] = 1$ it can be seen that $[y^3, x^3] = 1$.

Next we simplify (iii) and (iv). Notice that we can still use (v) and (vi) which are consequences of (i) and (ii). Write (iii) as

$$y^{-m}x^3y^{-m}x^{(n-1)/2}y^{-m}x^3y^{-m}x^{(n-1)/2} = 1$$

$$\Rightarrow \quad x^3yx^{(n-1)/2}yx^3yx^{(n-1)/2}y^{-m-3} = 1 \qquad \text{since } 3|(m+1)$$

$$(x^3yx^{(n-1)/2}y)^2 = y^{m+4}$$

Write (iv) as

$$x^ny^3x^ny^{-(m+1)/2}x^ny^3x^ny^{-(m+1)/2} = 1$$

$$\Rightarrow \quad x^{-2}y^3xy^{-(m+1)/2}xy^3xy^{-(m+1)/2}x^{n-1} = 1 \qquad \text{since } 3|(n-1)$$

$$(y^3xy^{-(m+1)/2}x)^2 = x^{4-n}.$$

We now write the relations of G as:

Theorem 7.2.1 G is generated by x and y subject to the relations

(.1.) $x^{3n} = 1$

(.2.) $y^{3m} = 1$

(.3.) $(xy^{-1})^2 = 1$

(.4.) $(x^3yx^{(n-1)/2}y)^2 = y^{m+4}$

(.5.) $(y^3xy^{-(m+1)/2}x)^2 = x^{4-n}$

(.6.) $x^{-n} = (y^mx)^2$

Lemma 7.2.5 In G we have $[x^n, yx^3y^{-1}] = [x^n, y^{-1}x^3y] = 1$, $[y^m, xy^3x^{-1}] = [y^m, x^{-1}y^3x] = 1$.

Proof: Since $y^m = x^{1-n}y^{-1}x^{1-n}y = yx^{1-n}y^{-1}x^{1-n}$ we have

$$[x^n, yx^{1-n}y^{-1}] = [x^n, y^{-1}x^{1-n}y] = 1$$

and cubing the second term in the commutators gives the result.

From (i)* we can deduce $x^n = y^{-m-1}xy^{-m-1}x^{-1} = x^{-1}y^{-m-1}xy^{-m-1}$.

We have $[y^m, xy^{-m-1}x^{-1}] = [y^m, x^{-1}y^{-m-1}x] = 1$ and cubing the second term in the commutators gives the result.

Lemma 7.2.6 Relations (.4.) and (.5.) in Theorem 7.2.1 can be replaced by

(.4.)* $(yx^{(n-1)/2}y^{-1}x^{-4})^2 = x^n$

(.5.)* $(xy^{(m+1)/2}x^{-1}y^4)^2 = y^m$

Proof: To obtain the new relation to replace (.4.) start from (iii)

$(y^{-m}x^3y^{-m}x^{(n-1)/2})^2 = 1$

$(y^{-m}x^3y^{-m-1}yx^{(n-1)/2})^2 = 1$

$(y^{-2m-1}x^3yx^{(n-1)/2})^2 = 1$ since $3|(-m-1)$

Use $y^{-2m-1} = x^{1-n}y^{-1}x^{1-n}$ to get $(y^{-1}x^{4-n}yx^{(-n+1)/2})^2 = 1$ so $yx^{(n-1)/2}y^{-1}x^{n-4}yx^{(n-1)/2}y^{-1}x^{n-4} = 1$. But $3|(n-1)/2$, so using Lemma 7.2.5 we have $yx^{(n-1)/2}y^{-1}x^{-4}yx^{(n-1)/2}y^{-1}x^{-4-n} = 1$ giving $(yx^{(n-1)/2}y^{-1}x^{-4})^2 = x^n$.

To obtain the new relation to replace (.5.) start from (iv)

$(x^ny^3x^ny^{-(m+1)/2})^2 = 1$

$(x^{2n-1}y^3xy^{-(m+1)/2})^2 = 1$

$(x^{-n-1}y^3xy^{-(m+1)/2})^2 = 1$

Use $x^{-n-1} = y^{m+1}x^{-1}y^{m+1}$ to get $(x^{-1}y^{m+4}xy^{(m+1)/2})^2 = 1$ so $xy^{(m+1)/2}x^{-1}y^{m+4}xy^{(m+1)/2}x^{-1}y^{m+4} = 1$. But $3|(m+1)/2$, so using Lemma 7.2.5 we have $(xy^{(m+1)/2}x^{-1}y^4) = y^m$.

Hence replacing the relations (.4.) and (.5.) respectively by (.4.)* and (.5.)* in Theorem 7.2.1 the presentation for G will be as in the following corollary.

Corollary 7.1

(I)	$x^{3n} = 1$	(IV)	$(yx^{(n-1)/2}y^{-1}x^{-4})^2 = x^n$
(II)	$y^{3m} = 1$	(V)	$(xy^{(m+1)/2}x^{-1}y^4)^2 = y^m$
(III)	$(xy^{-1})^2 = 1$	(VI)	$x^{-n} = (y^mx)^2$

The presentation given in Corollary 7. 1 is not efficient since $M(PSL(2,\mathbb{Z}_n) \times PSL(2,\mathbb{Z}_m)) = C_2 \times C_2$. However we conjecture:

Conjecture. For $n = 6k + 1$ and $m = 6t - 1$, $PSL(2,\mathbb{Z}_n) \times PSL(2,\mathbb{Z}_m)$ has the efficient presentation

$$G = \langle\, x,y \mid x^{3n} = 1,\ (xy^{-1})^2(yx^{(n-1)/2}y^{-1}x^{-4})^{-2} = x^{-n},$$
$$(xy^{(m+1)/2}x^{-1}y^4)^2 = y^m,\ x^{-n} = (y^m x)^2 \quad \rangle$$

(i) If $n \equiv 1(\mathrm{mod}6)$ and $m \equiv 1(\mathrm{mod}6)$ then replace m by $-m$ in the above presentation.

(ii) If $n \equiv -1(\mathrm{mod}6)$ and $m \equiv -1(\mathrm{mod}6)$ then replace n by $-n$ in the above presentation.

(iii) If $n \equiv -1(\mathrm{mod}6)$ and $m \equiv 1(\mathrm{mod}6)$ then replace n by $-n$ and m by $-m$ in the above presentation.

We have verified the conjecture for

(i) $n = 7,13,19,25,31,37,43,49$ and $m = 5$

(ii) $n = 49$ and $m = 7$

(iii) $n = 5,11,23,29,35,41,47$ and $m = 5$

which for the cases n= 25,m=5; n=49, n=7; n=35, m=5 the efficiency of G was previously not known.

(i);

Case n = 7, m = 5: Since $7 \equiv 1(\mathrm{mod}6)$ and $5 \equiv -1(\mathrm{mod}6)$, in Corollary 7.1 we have to replace n by 7 and m by 5. Using TC on subgroup $\langle x\rangle$ it can be seen that the relation (II) is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x\rangle$ it can be seen that the index of subgroup is 480. So the presentation for G is efficient. We only need to verify that x has order 21. Adding respectively $x^3 = 1$, $x^7 = 1$ we get respectively indexes 20, 24 for subgroup $\langle x\rangle$. So the order of x is 21.

Case n = 13, m = 5: Since $13 \equiv 1(\mathrm{mod}6)$ and $5 \equiv -1(\mathrm{mod}6)$, in Corollary 7.1 we have to replace n by 13 and m by 5. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (II) is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that the index of subgroup is 1680. So the presentation for G is efficient. We only need to verify that x has order 39. Adding respectively $x^3 = 1$, $x^{13} = 1$ we get respectively indexes 20, 84 for subgroup $\langle x \rangle$. So order of x is 39.

Case n = 19, m = 5: Since $19 \equiv 1(\mathrm{mod}6)$ and $5 \equiv -1(\mathrm{mod}6)$, in Corollary 7.1 we have to replace n by 19 and m by 5. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (II) is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that the index of subgroup is 3600. So the presentation for G is efficient. We only need to verify that x has order 57. Adding respectively $x^3 = 1$, $x^{19} = 1$ we get respectively indexes 20, 180 for subgroup $\langle x \rangle$. So the order of x is 57.

Case n = 25, m = 5: Since $25 \equiv 1(\mathrm{mod}6)$ and $5 \equiv -1(\mathrm{mod}6)$, in Corollary 7.1 we have to replace n by 25 and m by 5. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (II) is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that the index of subgroup is 6000. So the presentation for G is efficient. We only need to verify that x has order 75. Adding respectively $x^3 = 1$, $x^5 = 1$, $x^{15} = 1$, $x^{25} = 1$ we get respectively indexes 20, 12, 240, 300 for subgroup $\langle x \rangle$. So the order of x is 75.

Case n = 31, m = 5: Since $31 \equiv 1(\mathrm{mod}6)$ and $5 \equiv -1(\mathrm{mod}6)$, in Corollary 7.1 we have to replace n by 31 and m by 5. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (II) is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that the index of subgroup is 9600. So the presentation for G is efficient. We only need to verify that x has order 93. Adding respectively $x^3 = 1$, $x^{31} = 1$ we get respectively indexes 20, 480 for subgroup $\langle x \rangle$. So the order of x is 93.

Case n = 37, m = 5: Since $37 \equiv 1 (\text{mod} 6)$ and $5 \equiv -1 (\text{mod} 6)$, in Corollary 7.1 we have to replace n by 37 and m by 5. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (II) is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that the index of subgroup is 13680. So the presentation for G is efficient. We only need to verify that x has order 111. Adding respectively $x^3 = 1$, $x^{37} = 1$ we get respectively indexes 20, 924 for subgroup $\langle x \rangle$. So the order of x is 111.

Case n = 43, m = 5: Since $43 \equiv 1 (\text{mod} 6)$ and $5 \equiv -1 (\text{mod} 6)$, in Corollary 7.1 we have to replace n by 43 and m by 5. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (II) is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that the index of subgroup is 18480. So the presentation for G is efficient. We only need to verify that x has order 129. Adding respectively $x^3 = 1$, $x^{43} = 1$ we get respectively indexes 20, 924 for subgroup $\langle x \rangle$. So the order of x is 129.

Case n = 49, m = 5: Since $49 \equiv 1 (\text{mod} 6)$ and $5 \equiv -1 (\text{mod} 6)$, in Corollary 7.1 we have to replace n by 49 and m by 5. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (II) is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that the index of subgroup is 23520. So the presentation for G is efficient. We only need to verify that x has order 147. Adding respectively $x^3 = 1$, $x^7 = 1$, $x^{49} = 1$ we get respectively indexes 20, 24, 1176 for subgroup $\langle x \rangle$. So the order of x is 147.

Case n = 55, m = 5: Since $55 \equiv 1 (\mathrm{mod} 6)$ and $5 \equiv -1 (\mathrm{mod} 6)$, in Corollary 7.1 we have to replace n by 55 and m by 5. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (II) is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that the index of subgroup is 28800. So the presentation for G is efficient. We only need to verify that x has order 165. Adding respectively $x^3 = 1$, $x^5 = 1$, $x^{11} = 1$, $x^{15} = 1$, $x^{33} = 1$, $x^{55} = 1$ we get respectively indexes 20, 12, 60, 240, 1200, 1440 for subgroup $\langle x \rangle$. So the order of x is 165.

(ii);

Case n = 5, m = 5: Since $5 \equiv -1 (\mathrm{mod} 6)$ and $5 \equiv -1 (\mathrm{mod} 6)$, in Corollary 7.1 we have to replace n by -5 and m by 5. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (II) is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that the index of subgroup is 240. So the presentation for G is efficient. We only need to verify that x has order 15. Adding respectively $x^3 = 1$, $x^5 = 1$ we get respectively indexes 20, 12 for subgroup $\langle x \rangle$. So the order of x is 15.

Case n = 11, m = 5: Since $11 \equiv -1 (\mathrm{mod} 6)$ and $5 \equiv -1 (\mathrm{mod} 6)$, in Corollary 7.1 we have to replace n by -11 and m by 5. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (II) is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that the index of subgroup is 1200. So the presentation for G is efficient. We only need to verify that x has order 33. Adding respectively $x^3 = 1$, $x^{11} = 1$ we get respectively indexes 20, 60 for subgroup $\langle x \rangle$. So the order of x is 33.

Case n = 17, m = 5: Since $17 \equiv -1(\text{mod}6)$ and $5 \equiv -1(\text{mod}6)$, in Corollary 7.1 we have to replace n by -17 and m by 5. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (II) is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that the index of subgroup is 2880. So the presentation for G is efficient. We only need to verify that x has order 51. Adding respectively $x^3 = 1$, $x^{17} = 1$ we get respectively indexes 20, 144 for subgroup $\langle x \rangle$. So the order of x is 51.

Case n = 23, m = 5: Since $23 \equiv -1(\text{mod}6)$ and $5 \equiv -1(\text{mod}6)$, in Corollary 7.1 we have to replace n by -23 and m by 5. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (II) is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that the index of subgroup is 5280. So the presentation for G is efficient. We only need to verify that x has order 69. Adding respectively $x^3 = 1$, $x^{23} = 1$ we get respectively indexes 20, 264 for subgroup $\langle x \rangle$. So the order of x is 69.

Case n = 29, m = 5: Since $29 \equiv -1(\text{mod}6)$ and $5 \equiv -1(\text{mod}6)$, in Corollary 7.1 we have to replace n by -29 and m by 5. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (II) is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that the index of subgroup is 8400. So the presentation for G is efficient. We only need to verify that x has order 87. Adding respectively $x^3 = 1$, $x^{29} = 1$ we get respectively indexes 20, 420 for subgroup $\langle x \rangle$. So the order of x is 87.

Case n = 35, m = 5: Since $35 \equiv -1(\text{mod}6)$ and $5 \equiv -1(\text{mod}6)$, in Corollary 7.1 we have to replace n by -35 and m by 5. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (II) is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that the index of subgroup is 11520. So the presentation for G is efficient. We only need to verify that x has order 105. Adding respectively $x^3 = 1$, $x^5 = 1$, $x^7 = 1$, $x^{15} = 1$, $x^{21} = 1$, $x^{35} = 1$ we get respectively indexes 20, 12, 24, 240, 480, 576 for subgroup $\langle x \rangle$. So the order of x is 105.

Case n = 41, m = 5: Since $29 \equiv -1(\text{mod}6)$ and $5 \equiv -1(\text{mod}6)$, in Corollary 7.1 we have to replace n by -41 and m by 5. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (II) is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that the index of subgroup is 16800 So the presentation for G is efficient. We only need to verify that x has order 123. Adding respectively $x^3 = 1$, $x^{41} = 1$ we get respectively indexes 20, 840 for subgroup $\langle x \rangle$. So the order of x is 123.

Case n = 47, m = 5: Since $47 \equiv -1(\text{mod}6)$ and $5 \equiv -1(\text{mod}6)$, in Corollary 7.1 we have to replace n by -47 and m by 5. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (II) is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that the index of subgroup is 22080. So the presentation for G is efficient. We only need to verify that x has order 141. Adding respectively $x^3 = 1$, $x^{47} = 1$ we get respectively indexes 20, 1104 for subgroup $\langle x \rangle$. So the order of x is 141.

(iii);

Case n = 7, m = 7: Since $7 \equiv 1 (\mathrm{mod} 6)$, in Theorem 6.2.1 we have to replace n by 7 and m by -7. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (II) is redundant and combining relations (III) and (IV) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that the index of subgroup is 1344. So the presentation for G is efficient. We only need to verify that x has order 21. Adding respectively $x^3 = 1$, $x^7 = 1$ we get respectively indexes 56, 24 for subgroup $\langle x \rangle$. So the order of x is 21.

Case n = 49, m = 7: Since $49 \equiv 1 (\mathrm{mod} 6)$ and $7 \equiv 1 (\mathrm{mod} 6)$, in Theorem 6.2.1 we have to replace n by 49 and m by -7. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (.2.) is redundant and combining relations (.1.) and (.3.) and again using TC on subgroup $\langle x \rangle$ it can be seen that index of subgroup is 65856. So the presentation for G is efficient. We only need to verify that x has order 147. Adding respectively $a^3 = 1$, $a^7 = 1$, $a^{21} = 1$, $a^{49} = 1$, $a^{147} = 1$ we get respectively indexes 56, 24, 1344, 1176, 65856 for subgroup $\langle x \rangle$. So the order of x is 147.

Case n = 65, m = 5: Since $65 \equiv -1 (\mathrm{mod} 6)$ and $5 \equiv -1 (\mathrm{mod} 6)$, in Theorem 6.2.1 we have to replace n by -65 and m by 5. Using TC on subgroup $\langle x \rangle$ it can be seen that the relation (.2.) is redundant and combining relations (.1.) and (.3.) and again using TC on subgroup $\langle x \rangle$ it can be seen that index of subgroup is 40320. So the presentation for G is efficient. We only need to verify that x has order 195. Adding respectively $a^3 = 1$, $a^5 = 1$, $a^{13} = 1$, $a^{15} = 1$, $a^{39} = 1$, $a^{65} = 1$, $a^{195} = 1$ we get respectively indexes 20, 12, 84, 240, 1680, 2016, 40320 for subgroup $\langle x \rangle$. So the order of x is 195.

7.3 THE GROUPS $G = PSL(2,\mathbb{Z}_p) \times PSL(2,3^2)$:

Let $G = PSL(2,\mathbb{Z}_p) \times PSL(2,3^2)$. Then, using a presentation for $PSL(2,\mathbb{Z}_p)$ given in [36] and using a presentation for $PSL(2,3^2)$ given in [21].

$$G = \langle\, a,b,c,d \mid a^2 = b^p = (ab)^3 = (ab^4ab^{(p+1)/2})^2 = 1,\ c^2 = d^4 = (cd)^5 = (cd^2)^5 = 1,$$
$$[a,c] = [a,d] = [b,c] = [b,d] = 1 \quad \rangle.$$

Put $x = bcd$, $y = abd$. Then let $p = 10n + 1$. We have $x^5 = b^5 \Rightarrow x^{p-1} = b^{p-1} = b^{-1}$ so $b = x^{1-p}$. Similarly $y^3 = d^3 \Rightarrow y^3 = d^3 = d^{-1}$ so $d = y^{-3}$. Since $x = bcd \Rightarrow c = x^py^3$. Also since $y = abd \Rightarrow a = y^4x^{p-1}$. We have proved:

Lemma 7.3.1 If $x = bcd$, $y = abd$ then $a = y^4x^{p-1}$, $b = x^{1-p}$, $c = x^py^3$, $d = y^{-3}$.

We write down the 12 relations of G written in terms of x and y in the order they appear in the presentation above

(1) $(y^4x^{p-1})^2 = 1$

(2) $(x^{1-p})^p = 1$

(3) $y^{12} = 1$

(4) $(y^4x^{p-1}(x^{1-p})^4y^4x^{p-1}(x^{1-p})^{(p+1)/2})^2 = 1$

(5) $(x^py^3)^2 = 1$

(6) $y^{12} = 1$

(7) $x^{5p} = 1$

(8) $(x^py^{-3})^5 = 1$

(9) $[y^4x^{p-1}, x^py^3] = 1$

(10) $[y^4x^{p-1}, y^{-3}] = 1$

(11) $[x^{1-p}, x^py^3] = 1$

(12) $[x^{1-p}, y^{-3}] = 1$

Lemma 7.3.2 In lemma 7.3.1 the relations (2),(6),(10),(11) are redundant.

Proof: Since $5|(1-p)$ then x^{1-p} is a power of x^5 so, since $x^{5p}=1$ we have $(x^{1-p})^p = 1$. (6) is identical to (3). Also (10) and (11) are immediate consequences of (12).

We now tidy up a bit. Since $[x^{1-p},y^{-3}] = 1$, taking 5th power of first term we can get $[x^5,y^3] = 1$. We now have presentation for G as follows.

Lemma 7.3.3 G is generated by x and y subject to the relations

(i) $(y^4x^{p-1})^2 = 1$	(v) $[y^4,x^p] = 1$
(ii) $(x^py^3)^2 = 1$	(vi) $[x^5,y^3] = 1$
(iii) $(x^py^{-3})^5 = 1$	(vii) $x^{5p} = 1$
(iv) $(y^4x^{3-3p}y^4x^k)^2 = 1$	(viii) $y^{12} = 1$

where $k = (1-p)((p+1)/2) + (p-1)$.

Consider (i). We have $y^4x^{p-1}y^4x^{p-1} = 1$ and using (v) and (vii) this gives $x^{2p-1}y^4x^{-1}y^4 = 1$. Hence $x^{2p-1} = y^{-4}xy^{-4}$ replaces (i). Similarly consider (ii). We have $x^py^3x^py^3 = 1$ and using (v) and (viii) this gives $y^7x^py^{-1}x^p = 1$. Hence $y^7 = x^{-p}yx^{-p}$ replaces (ii). But consider again $y^4x^{p-1}y^4x^{p-1} = 1$ and this time use (vi) in the form $[x^5,y^3] = 1$. We have $y^7x^{p-1}yx^{p-1} = 1$ and substituting $y^7 = x^{-p}yx^{-p}$ into this relation gives $x^{-p}yx^{-p}x^{p-1}yx^{p-1} = 1$ so $(xy^{-1})^2 = 1$. This now replaces $y^7 = x^{-p}yx^{-p}$. Use this to replace $x^{2p-1} = y^{-4}xy^{-4}$ by $x^{2p-1} = y^{-3}x^{-1}y^{-3}$. We have new relations (i)* and (ii)* to replace respectively (i) and (ii). They are

(i)* $x^{2p-1} = y^{-3}x^{-1}y^{-3}$

(ii)* $(xy^{-1})^2 = 1$

Lemma 7.3.4 $[y^4,x^p] = 1$ and $[x^5,y^3] = 1$ are redundant.

Proof: From Lemma 7.3.3 we have $x^{2p-1} = y^{-4}xy^{-4}$ therefore $x^{2p} = (y^{-4}x)^2$ so $[y^{-4}x,x^{2p}] = 1$. Cubing the second term of commutator i.e. $[y^{-4}x,x^{6p}] = 1$. But $x^{5p} = 1 \Rightarrow [y^{-4}x,x^p] = 1 \Rightarrow [y^4,x^p] = 1$.

Consider (i)* i.e. $x^{2p-1} = y^{-3}x^{-1}y^{-3} \Rightarrow x^{2p-2} = y^{-3}x^{-1}y^{-3}x^{-1}$

$\Rightarrow x^{2p-2} = (y^{-3}x^{-1})^2 \Rightarrow [y^{-3},x^{2p-2}] = 1$.

Raising the second term of $[y^3,x^{2p-2}] = 1$ to the power 5 and using (vii) we have $[y^3,x^{10}] = 1$. Consider again $[y^3,x^{2p-2}] = 1$ and using (vii) we have $[y^3,x^{3p+2}] = 1$. Cubing the second term in the commutator gives $[y^3,x^{9p+6}] = 1$. Using (vii) we have $[y^3,x^{-p+6}] = 1$, since $p = 10n+1$ and since $[y^3,x^{10}] = 1 \Rightarrow [y^3,x^{-p+6}] = [y^3,x^5] = 1$.

Next we simplify (iii) and (iv). Notice that we can still use (v) and (vi) which are consequences of (i) and (ii).

Consider (iii). $(x^py^{-3})^5 = 1 \Rightarrow (xx^{p-1}y^{-3})^5 = 1$ since $[y^3,x^{p-1}] = 1$

$\Rightarrow (xy^{-3})^5x^{5p-5} = 1$ using (vii)

$\Rightarrow (xy^{-3})^5 = x^5$ this replaces (iii).

Consider (iv).

$(y^4x^{3-3p}y^4x^k)^2 = 1$ since $[x^p,y^4] = 1$

$\Rightarrow (y^4x^3y^4x^{k-3p})^2 = 1$ since $k = (1-p)((p+1)/2) + (p-1)$

$\Rightarrow (y^4x^3y^4x^{(1-p)((p+1)/2)-1})^2x^{-4p} = 1$

$\Rightarrow (y^4x^3y^4x^{(1-p)((p+1)/2)-1})^2x^p = 1$ use (vii)

since $p = 5n+1$ and p is odd $\Rightarrow 5|(1-p)/2 \Rightarrow x^{p(1-p)/2} = 1$ using (vii)

$(y^4x^3y^4x^{-(p+1)/2)})^2x^p = 1$ replaces (iv). We now write down the relations of G as:

Theorem 7.3.1 G is generated by x and y subject to

(.1.) $x^{5p} = 1$

(.2.) $y^{12} = 1$

(.4.) $(y^4x^3y^4x^{-(p+1)/2)})^2 = x^{-p}$

(.5.) $(xy^{-1})^2 = 1$

(.3.) $(xy^{-3})^5 = x^5$ (.6.) $x^{2p-1} = y^{-3}x^{-1}y^{-3}$

The presentation given in Theorem 7.3.1 is not efficient since $M(PSL(2,\mathbb{Z}_p) \times PSL(2,3^2)) = C_2 \times C_2$. However we conjecture:

Conjecture. For $p = 10n + 1$ the groups $PSL(2,\mathbb{Z}_p) \times PSL(2,3^2)$ have the efficient presentation

$$G = \langle\, x,y \mid x^{5p} = 1, (xy^{-1})^2 = 1,\ x^{2p-1} = y^{-3}x^{-1}y^{-3},$$
$$(xy^{-3})^5 x^{p-5}(y^4x^3y^4x^{-(p+1)/2})^2 = 1 \,\rangle$$

If $p \equiv -1 \pmod{10}$ replace p by $-p$.

We have verified the conjecture for $p = 9,11,19$ and 21 which in these cases the efficiency of G was previously not known.

Case $p = 9$: Since $9 \equiv -1 \pmod{10}$, in Theorem 7.3.1 we have to replace p by -9. Using TC on the subgroup $\langle x \rangle$ it can be seen that the relation (.2.) is redundant and combining relations (.3.) and (.4.) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that index of subgroup is 2592. So the presentation for G is efficient. We only need to verify that x has order 45. Adding respectively $x^3 = 1$, $x^5 = 1$, $x^{15} = 1$ we get respectively indexes 4, 72, 288 for subgroup $\langle x \rangle$. So the order of x is 45.

Case $p = 11$: Since $11 \equiv 1 \pmod{10}$, in Theorem 7.3.1 we have to replace p by 11. Using TC on the subgroup $\langle x \rangle$ it can be seen that the relation (.2.) is redundant and combining relations (.3.) and (.4.) as in the conjecture and again using TC on the subgroup $\langle x \rangle$ it can be seen that index of subgroup is 4320. So the presentation for G is efficient. We only need to verify that x has order 55. Adding respectively $x^5 = 1$, $x^{11} = 1$, we get respectively indexes 72, 60 for subgroup $\langle x \rangle$. So the order of x is 55.

Case p = 19: Since $19 \equiv -1(\text{mod}10)$, in Theorem 7.3.1 we have to replace p by -19. Using TC on the subgroup $\langle x \rangle$ it can be seen that the relation (.2.) is redundant and combining relations (.3.) and (.4.) as in the conjecture and again using TC on subgroup $\langle x \rangle$ it can be seen that index of subgroup is 12960. So the presentation for G is efficient. We only need to verify that x has order 95. Adding respectively $x^5 = 1$, $x^{19} = 1$ we get respectively indexes 72, 180 for subgroup $\langle x \rangle$. So the order of x is 95.

Case p = 21: Since $21 \equiv 1(\text{mod}10)$, in Theorem 7.3.1 we have to replace p by 21. Using TC on the subgroup $\langle x \rangle$ it can be seen that the relation (.2.) is redundant and combining relations (.3.) and (.4.) as in the conjecture and again using TC on the subgroup $\langle x \rangle$ it can be seen that index of subgroup is 13824. So the presentation for G is efficient. We only need to verify that x has order 105. Adding respectively $x^3 = 1$, $x^5 = 1$, $x^7 = 1$, $x^{15} = 1$, $x^{21} = 1$ we get respectively indexes 4, 72, 24, 288, 192 for subgroup $\langle x \rangle$. So the order of x is 105.

7.4 THE EFFICIENCY OF $PSL(2,5) \times PSL(2,3^2)$, $PSL(2,7) \times PSL(2,3^2)$ and $PSL(2,5) \times PSL(2,3^3)$:

Theorem 7.4.1 Let $G = PSL(2,5) \times PSL(2,3^2)$, where $PSL(2,5)$ and $PSL(2,3^2)$ are simple groups, then G has the efficient presentation

$$G = \langle\, x,y \mid x^{15} = 1,\ (xy^{-1})^2 = 1,\ x^{11} = y^4xy^4,\ (x^6y^5)^5 = 1 \,\rangle .$$

Proof: Let $G = PSL(2,5) \times PSL(2,3^2)$. Then using the presentations for $PSL(2,5)$ and $PSL(2,3^2)$ given in [21],

$$G = \langle\, a,b,c,d \mid a^2 = b^3 = (ab)^5 = 1,\ c^2 = d^4 = (cd)^5 = (cd^2)^5 = 1,\ [a,c] = [a,d] = [b,c] = [b,d] = 1 \,\rangle$$

Put $x = bcd$, $y = abd$. We have

$x^5 = b^5$ so $b = x^{-5}$. Similarly $y^5 = d^5$ so $d = y^5$. Since $x = bcd \Rightarrow c = x^6y^{-5}$. Also since $y = abd$ then $a = y^{-4}x^5$.

The relations of G, written in terms of x and y, simplifying and eliminating obviously redundant relations are:

(i)	$(y^{-4}x^5)^2 = 1$	(v)	$[x^5,y^5] = 1$
(ii)	$(x^6y^{-5})^2 = 1$	(vi)	$x^{15} = 1$
(iii)	$(x^6y^5)^5 = 1$	(vii)	$y^{20} = 1$
(iv)	$[x^6,y^4] = 1$		

Consider (i). We have $y^{-4}x^5y^{-4}x^5 = 1$ and using (iv) this gives $x^{11}y^{-4}x^{-1}y^{-4} = 1$. Hence $x^{11} = y^4xy^4$ replaces (i). Similarly consider (ii). We have $x^6y^{-5}x^6y^{-5} = 1$ and using (iv) this gives $y^{-1}x^6y^{-9}x^6 = 1$. Hence $y^9 = x^6y^{-1}x^6$ replaces (ii). But consider again $y^{-4}x^5y^{-4}x^5 = 1$ and this time use (v) in the form $[x^5,y^5] = 1$. We have $y^{-9}x^5yx^5 = 1$ and substituting $y^9 = x^6y^{-1}x^6$ into this relation gives $x^{-6}yx^{-6}x^5yx^5 = 1$ so $(x^{-1}y)^2 = 1$. This now replaces $y^9 = x^6y^{-1}x$. We have new relations (i)* and (ii)* to replace respectively (i) and (ii). They are

(i)* $x^{11} = y^4xy^4$

(ii)* $(x^{-1}y)^2 = 1$

The next step is to show that $[x^5,y^5] = 1$ and $[x^6,y^4] = 1$ are redundant. Using the (ii)* we can rewrite (i)* as $x^{10} = (y^5x^{-1})^2$. So $[y^5x^{-1},x^{10}] = 1 \Rightarrow [y^5,x^{10}] = 1$. Since the order of x is 15 and squaring the second term of the commutator we get $[y^5,x^5] = 1$. From (i)* we get $x^{10} = (y^4x)^2$ so $[y^4,x^{12}] = 1$. Order of x is 15 so $[x^3,y^4] = 1 \Rightarrow [x^6,y^4] = 1$.

At this stage we have G generated by x and y subject to the five relations:

(1)	$x^{15} = 1$	(4)	$(x^6y^5)^5 = 1$

(2) $y^{20} = 1$ (5) $x^{11} = y^4xy^4$

(3) $(x^{-1}y)^2 = 1$

Using TC it can be verified that $y^{20} = 1$ is redundant. Using TC on the subgroup $\langle x\rangle$, we can verify that the presentation for G given in Theorem 7.4.1 is efficient. The index of $\langle x\rangle$ in G is 1440. It has to be checked whether the order of x is not a proper divisor of 15. It can be verified easily that the order of x is 15, because if x has order 3 then, using TC, it can be seen that G collapses to PSL(2,5), which is not the case. If x has order 5 then using TC on the subgroup $\langle x\rangle$ it can be seen that G is PSL(2,3^2), which is not the case. So the order of x must be 15.

Theorem 7.4.2 Let G = PSL(2,7) × PSL(2,3^2), where PSL(2,7) and PSL(2,3^2) are simple groups. Then G has the efficient presentation

$$G = \langle\, x,y \mid y^{28} = 1,\ (yx^{-1})^2 = 1,\ x^4 = y^8x^{-1}y^8,\ (x^6y)^5y^{-12}(x^5y^{-2})^4 = 1 \,\rangle.$$

Proof: Let G = PSL(2,7) × PSL(2,3^2). Then using the presentations for PSL(2,7) and PSL(2,3^2) given in [21],

$$G = \langle\, a,b,c,d \mid a^2 = b^3 = (ab)^7 = 1,\ [a,b]^4 = 1,\ c^2 = d^4 = (cd)^5 = (cd^2)^5 = 1,\ [a,c] = [a,d] = [b,c] = [b,d] = 1 \,\rangle$$

Put $x = bcd$, $y = abd$. We have $x^5 = b^5$ and $b^2 = b^{-1} \Rightarrow b = x^{-5}$. Similarly $y^7 = d^7$ and $y^7 = y^{-1} \Rightarrow d = y^{-7}$. Since $x = bcd \Rightarrow c = x^6y^7$. Also since $y = abd$ then $a = y^8x^5$.

The relations of G, written in terms of x and y, simplifying and eliminating obviously redundant relations, are:

(i) $(y^8x^5)^2 = 1$ (v) $[x^5,y^7] = 1$

(ii) $(x^6y^7)^2 = 1$ (vi) $[y^8,x^6] = 1$

(iii) $(x^6y^{-7})^5 = 1$ (vii) $y^{28} = 1$

(iv) $[y^8x^5, x^{-5}]^4 = 1$ (viii) $x^{15} = 1$

Consider (i). We have $y^8x^5y^8x^5 = 1$ and using (vi) this gives $x^{11}y^8x^{-1}y^8 = 1$. Hence $x^{-11} = y^8x^{-1}y^8$ replaces (i). Similarly consider (ii). We have $x^6y^7x^6y^7 = 1$ and using (vi) this gives $y^{-1}x^6y^{15}x^6 = 1$. Hence $y^{-15} = x^6y^{-1}x^6$ replaces (ii). But consider again $y^8x^5y^8x^5 = 1$ and this time use (v) in the form $[x^5, y^7] = 1$. We have $y^{15}x^5yx^5 = 1$ and substituting $y^{-15} = x^6y^{-1}x^6$ into this relation gives $(x^{-1}y)^2 = 1$. This now replaces $y^{-15} = x^6y^{-1}x^6$. We have new relations (i)* and (ii)* to replace respectively (i) and (ii). They are

(i)* $x^4 = y^8x^{-1}y^8$

(ii)* $(x^{-1}y)^2 = 1$

The next step is to show that $[x^5, y^7] = 1$ and $[y^8, x^6] = 1$ are redundant. Using the new (ii)* we can rewrite (i)* as $x^5 = (y^7x)^2$. So $[y^7x, x^5] = 1 \Rightarrow [y^7, x^5] = 1$.

Also using (ii)* we can rewrite (i)* as $x^3 = (y^8x^{-1})^2 \Rightarrow [y^8, x^3] = 1$

Squaring second term of $[y^8, x^3]$ gives $[y^8, x^6] = 1$.

At this stage we have G generated by x and y subject to the six relations:

(1) $x^{15} = 1$ (4) $(x^6y^{-7})^5 = 1$

(2) $y^{28} = 1$ (5) $[y^8x^5, x^{-5}]^4 = 1$

(3) $(x^{-1}y)^2 = 1$ (6) $x^4 = y^8x^{-1}y^8$

Next we simplify (4) and (5). Notice that we can still use (v) and (vi) which are consequences of (i) and (ii). Consider (4).

$(x^6y^{-7})^5 = 1 \Rightarrow (x^6y)^5y^{-40} = 1$ since $[y^8, x^6] = 1$

$\Rightarrow (x^6y)^5 = y^{12}$ using (2)

$\Rightarrow (x^6y)^5 = y^{12}$ and this replaces (4).

Write (5) as $(y^8x^{-5}y^{-8}x^5)^4 = 1$

$(y^8xy^{-8}x^{-1})^4 = 1$ using $[x^6, y^8] = 1$

$\Rightarrow (x^5y^{-2})^4 = 1$ using $[x^5, y^7] = 1$ and (6)

$(x^5y^{-2})^4 = 1$ and this replaces (5).

At this stage we have G generated by x and y subject to the six relations:

(1) $x^{15} = 1$	(4) $(x^6y)^5 = y^{12}$
(2) $y^{28} = 1$	(5) $(x^5y^{-2})^4 = 1$
(3) $(x^{-1}y)^2 = 1$	(6) $x^4 = y^8x^{-1}y^8$

Using TC it can be verified that $x^{15} = 1$ is redundant. Combining relation (4) with relation (5) and using TC on the subgroup $\langle y \rangle$ we can verify that the resulting presentation for G is efficient. The index of $\langle y \rangle$ in G is 2160. It has to be checked that the order of y is 28. It can be verified that the order of y is 28, because if y had order 7 or 14 then G is PSL(2,7), if y had order 4 then G is PSL$(2,3^2)$ and if y had order 2 then G is trivial group. So y must have order 28.

Theorem 7.4.3 Let G = PSL(2,5) × PSL$(2,3^3)$, where PSL(2,5) and PSL$(2,3^3)$ are simple groups, then G has the efficient presentation

$$G = \langle\, x,y \mid x^{39}(yx^{-1})^2 = 1,\ x^{-13} = (y^5x)^2,\ (x^3(xy^{-5})^3)^2 = 1,\ y^{15} = 1 \,\rangle .$$

Proof: Let G = PSL(2,5) × PSL$(2,3^3)$. Then using the presentations for PSL(2,5) and PSL$(2,3^3)$ given in [21],

$$G = \langle\, a,b,c,d \mid a^2 = b^3 = (ab)^5 = 1,\ c^2 = d^3 = (cd)^{13} = ((cd)^3(cd^{-1})^3)^2 = 1,$$
$$[a,c] = [a,d] = [b,c] = [b,d] = 1 \,\rangle$$

Put x = bcd, y = abd. We have

$x^{13} = b^{13}$ so $b = x^{13}$. Similarly $y^5 = d^5 = d^{-1}$ so $d = y^{-5}$. Since $x = bcd \Rightarrow$ $c = x^{-12}y^5$. Also since $y = abd \Rightarrow a = y^6x^{-13}$.

The relations of G, written in terms of x and y, simplifying and eliminating obviously redundant relations, are:

(i) $(y^6x^{-13})^2 = 1$	(v) $[x^{13},y^5] = 1$

(ii) $(x^{-12}y^5)^2 = 1$ (vi) $x^{39} = 1$

(iii) $(x^3(xy^{-5})^3)^2 = 1$ (vii) $y^{15} = 1$

(iv) $[x^{-12},y^6] = 1$

Consider (i). We have $y^6x^{-13}y^6x^{-13} = 1$ and using (iv) this gives $x^{-25}y^6x^{-1}y^6 = 1$. Hence $x^{25} = y^6x^{-1}y^6$ replaces (i). Similarly consider (ii). We have $x^{-12}y^5x^{-12}y^5 = 1$ and using (iv) this gives $y^{-11}x^{12}y^{-1}x^{12} = 1$. Hence $y^{11} = x^{12}y^{-1}x^{12}$ replaces (ii). But consider again $y^6x^{-13}y^6x^{-13} = 1$ and this time use (v) in the form $[x^{13},y^5] = 1$. We have $y^{11}x^{-13}yx^{-13} = 1$ and substituting $y^{11} = x^{12}y^{-1}x^{12}$ into this relation gives $(x^{-1}y)^2 = 1$. This now replaces $y^{11} = x^{12}y^{-1}x^{12}$. We have new relations (i)* and (ii)* to replace respectively (i) and (ii). They are

(i)* $x^{25} = y^6x^{-1}y^6$

(ii)* $(x^{-1}y)^2 = 1$.

The next step is to show that $[x^{13},y^5] = 1$ and $[x^{12},y^6] = 1$ are redundant. Using the new (ii)* and (vi) we can rewrite (i)* as $x^{-13} = (y^5x)^2$. So $[y^5x,x^{-13}] = 1 \Rightarrow$ $[y^5,x^{-13}] = 1 \Rightarrow [y^5,x^{13}] = 1$.

Also using (i)* we can get $x^{24} = (y^6x^{-1})^2 \Rightarrow [y^6,x^{24}] = 1$.

Raising the second term of $[y^6,x^{24}] = 1$ to the power six and using (vi) we have $[x^{12},y^6] = 1$. At this stage we have G generated by x and y subject to the following five relations:

(1) $x^{39} = 1$ (4) $(x^3(xy^{-5})^3)^2 = 1$

(2) $y^{15} = 1$ (5) $x^{-13} = (y^5x)^2$

(3) $(x^{-1}y)^2 = 1$

Let $G^* = \langle x,y \mid x^{39}(yx^{-1})^2 = 1,\ x^{-13} = (y^5x)^2,\ (x^3(xy^{-5})^3)^2 = 1, y^{15} = 1 \rangle$. From relation 1 in G^* it can be easily seen that x^{39} commutes with y. On the other hand G^* is perfect so $x^{39} \in Z(G^*) \cap G^{*\prime}$. Now consider $G^*/\langle x^{39} \rangle \cong G$. Since $x^{39} \in Z(G^*) \cap G^{*\prime}$ we can deduce $\langle x^{39} \rangle \leq M(G) = C_2 \times C_2$. This means

$| \langle x^{39} \rangle | = 1$ or 2 . Using TC on the subgroups $\langle x \rangle$ and $\langle x^2 \rangle$ it can be seen that $\langle x^{39} \rangle$ is a trivial group. So $G^* \cong G$ i.e. we have obtained the efficient presentation for G as claimed in the Theorem 7.4.3 .

Additionally in this section we shall investigate the structure of the following perfect group. In [32] Sandlobes classified all perfect groups of order up to 10^4. The following perfect group has order 161280 so its structure has not been shown previously.

Let $G = \langle a,b \mid a^5 = 1,\ b^7 = 1,\ (ab)^3 = 1,\ (b^2a^{-2})^2 = 1 \rangle$.

Abelianising the relations of G it can be seen that G is a perfect group. Using TC on subgroup $\langle b \rangle$ it can be seen that order of G is 161280. Now adding $(ab^{-1})^5 = 1$ to the presentation of G we can get homomorphic image of order 2520. Let us denote the homomorphic image by G^*. Using the CAYLEY program 3 it can be verified that G^* is A_7 .

<u>CAYLEY program 3:</u>

```
>cay;
>set workspace=2000000;
>G*:free(a,b);
>G*.relations: a^5, b^7, (a*b)^3, (b^2*a^-2)^2, (a*b^-1)^5 ;
>h = < b >;
>f,i,k = cosact homomorphism(G*,h);
>PRINT composition factor(i);
>QUIT;
```

Now consider G and the subgroup

$K = \langle k_1 = (ab^{-1})^5, k_2 = b(ab^{-1})^5b^{-1}, k_3 = ab(ab^{-1})^5b^{-1}a^{-1}, k_4 = ba(ab^{-1})^5a^{-1}b^{-1}, k_5 = ab^2(ab^{-1})^5b^{-2}a^{-1}, k_6 = a^2b(ab^{-1})^5a^{-2}b^{-1} \rangle$. Using TC it can be verified that index of K in G is 2520. Since order of G is 161280, the order of K is 64. Using the following CAYLEY program 4, outlined below, we can see that actually K is isomorphic to the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

With the CAYLEY program 4 we check that the generators of K are of order 2 and that they commute with each other.

CAYLEY program 4:

```
>cay;
>set workspace=2000000;
>G:free(a,b);
>G.relations: a^5, b^7, (a*b)^3, (b^2*a^-2)^2 ;
>q = SEQ((a*b^-1)^5, b*(a*b^-1)^5*b^-1, a*b*(a*b^-1)^5*b^-1*a^-1,
b*a*(a*b^-1)^5*a^-1*b^-1, a*b^2*(a*b^-1)^5*b^-2*a^-1,
a^2*b*(a*b^-1)^5*b^-1*a^-2);
>FOR  i TO 6 BY 1 DO
>PRINT order(q[i]);
>END;
>FOR i TO 6 DO
>  FOR j TO 6 DO
>     PRINT order((q[i]*q[j])^2);
>  END;
>END;
>QUIT;
```

Therefore G has A_7 and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as homomorphic images.

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